THE STEINBERG RELATION

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We give a short proof of the Steinberg relation in unstable motivic homotopy theory:

Theorem. *Let S be a scheme. The suspension of the Steinberg map*

$$\mathfrak{st}\colon (\mathbb{A}^1 \smallsetminus \{0,1\})_+ \to \mathbb{G}_m^{\wedge 2}, \quad a \mapsto (a, 1-a),$$

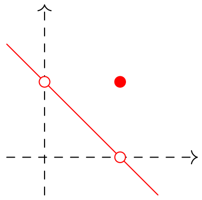
becomes nullhomotopic in $\mathcal{H}_\bullet(S)$. In fact, $\mathrm{L}_{\mathrm{Zar},\mathbb{A}^1}\Sigma(\mathfrak{st}) \simeq 0$.

If $a \in \mathcal{O}(S)$ is such that a and $1-a$ are invertible, it follows that the composition

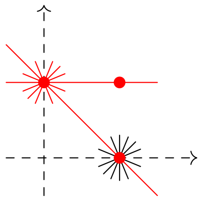
$$\mathrm{S}^1 = \Sigma(S_+) \xrightarrow{a} \Sigma((\mathbb{A}^1 \smallsetminus \{0,1\})_+) \xrightarrow{\mathfrak{st}} \Sigma\mathbb{G}_m^{\wedge 2}$$

is nullhomotopic in $\mathcal{H}_\bullet(S)$, i.e., the Steinberg relation holds in $[\mathrm{S}^1, \Sigma\mathbb{G}_m^{\wedge 2}]_{\mathcal{H}_\bullet(S)}$.

This result was first claimed by Hu and Kriz [HK01], but as pointed out by Druzhinin [Dru18] their proof is flawed. Indeed, Hu and Kriz prove the weaker statement that the suspension of the map $\mathbb{A}^1 \smallsetminus \{0,1\} \to \mathbb{G}_m^{\wedge 2}$ is nullhomotopic, which does not imply any relations in $[\mathrm{S}^1, \Sigma\mathbb{G}_m^{\wedge 2}]_{\mathcal{H}_\bullet(S)}$. In [Dru18], Druzhinin proves the Steinberg relation in *stable* motivic homotopy theory. Our argument below follows that of Hu and Kriz with one small but essential modification: instead of extending the Steinberg embedding from $\mathbb{A}^1 \smallsetminus \{0,1\}$ to $\mathbb{A}^1$, we extend it from $(\mathbb{A}^1 \smallsetminus \{0,1\})_+$ to a chain of three affine lines C, which is still $\mathbb{A}^1$-contractible:



The Steinberg embedding
$(\mathbb{A}^1 \smallsetminus \{0,1\})_+ \subset \mathbb{G}_m \times \mathbb{G}_m$



The extended Steinberg embedding
$C \subset \mathrm{Bl}_{\{(0,1),(1,0)\}}(\mathbb{A}^2) \smallsetminus ((\mathbb{A}^1 \times 0) \cup (0 \times \mathbb{A}^1))$

Proof. Let B be the blowup of $\mathbb{A}^2$ at the points $(0,1)$ and $(1,0)$ with the strict transforms of the coordinate axes removed:

$$B = \mathrm{Bl}_{\{(0,1),(1,0)\}}(\mathbb{A}^2) \smallsetminus ((\mathbb{A}^1 \times 0) \cup (0 \times \mathbb{A}^1)).$$

The open subschemes

$$U = \mathrm{Bl}_{(0,1)}(\mathbb{A}^1 \times \mathbb{G}_m) \smallsetminus (0 \times \mathbb{G}_m) \quad \text{and} \quad U' = \mathrm{Bl}_{(1,0)}(\mathbb{G}_m \times \mathbb{A}^1) \smallsetminus (\mathbb{G}_m \times 0)$$

form an open covering of B with intersection $\mathbb{G}_m \times \mathbb{G}_m$. Let

$$e\colon (\mathbb{G}_m \times \mathbb{G}_m) \sqcup_{\mathbb{G}_m \times 1} (\mathbb{A}^1 \times 1) \hookrightarrow U \quad \text{and} \quad e'\colon (\mathbb{G}_m \times \mathbb{G}_m) \sqcup_{1 \times \mathbb{G}_m} (1 \times \mathbb{A}^1) \hookrightarrow U'$$

be the obvious embeddings. Since $(\mathbb{G}_m \times \mathbb{A}^1) \sqcup_{\mathbb{G}_m \times 1} (\mathbb{A}^1 \times 1)$ and $\mathrm{Bl}_{(0,1)}(\mathbb{A}^2) \smallsetminus (0 \times \mathbb{A}^1) \simeq \mathbb{A}^2$ are $\mathbb{A}^1$-contractible, $\mathrm{L}_{\mathbb{A}^1}\Sigma e$ can be identified with $\mathrm{L}_{\mathbb{A}^1}$ of the embedding

$$(\mathbb{G}_m \times \mathbb{A}^1)/(\mathbb{G}_m \times \mathbb{G}_m) \hookrightarrow (\mathrm{Bl}_{(0,1)}(\mathbb{A}^2) \smallsetminus (0 \times \mathbb{A}^1))/U,$$

which is obviously a Zariski equivalence. Hence, Σe and $\Sigma e'$ are $\mathrm{L}_{\mathrm{Zar},\mathbb{A}^1}$-equivalences.

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Let $C \subset B$ be the closed subscheme composed of the following three affine lines, as depicted in the above figure: the line joining $(1,1)$ to $(0,1)$, the exceptional divisor over $(0,1)$, and the line joining $(0,1)$ to $(1,0)$. Note that C is $\mathbb{A}^1$-contractible. We then have a commutative diagram

$$\begin{tikzcd}
(\mathbb{A}^1 \smallsetminus \{0,1\})_+ \arrow[r, hook, "\mathfrak{st}"] \arrow[dd, hook, dashed] & \mathbb{G}_m \times \mathbb{G}_m \arrow[d, hook] & \\
& (\mathbb{A}^1 \times 1) \sqcup_{\mathbb{G}_m \times 1} (\mathbb{G}_m \times \mathbb{G}_m) \sqcup_{1 \times \mathbb{G}_m} (1 \times \mathbb{A}^1) \arrow[r, "\simeq_{\mathbb{A}^1}"] \arrow[d, hook, "e \sqcup e'"] & \mathbb{G}_m \wedge \mathbb{G}_m \\
{* \simeq_{\mathbb{A}^1} C} \arrow[r, hook] & \mathrm{L}_{\mathrm{Zar}}(U \sqcup_{\mathbb{G}_m \times \mathbb{G}_m} U') = B. &
\end{tikzcd}$$

Since the map $e \sqcup e'$ becomes an $\mathrm{L}_{\mathrm{Zar},\mathbb{A}^1}$-equivalence after one suspension, the theorem is proved. ☐

References

[Dru18] A. Druzhinin, *The homomorphism of presheaves* $\mathrm{K}_*^{\mathrm{MW}} \to \pi_s^{*,*}$ *over a base*, 2018, arXiv:1809.00087v3

[HK01] P. Hu and I. Kriz, *The Steinberg relation in* $\mathbb{A}^1$*-stable homotopy*, Int. Math. Res. Not. **2001** (2001), no. 17, pp. 907–912