COHERENT SIX OPERATIONS

MARC HOYOIS

I describe a symmetric monoidal $(\infty, 2)$-category Ξ such that a right-lax symmetric monoidal functor

$$\mathrm{D}\colon \Xi \to \mathrm{Pr}^{\mathrm{L}}$$

encodes a system of coefficients on derived schemes with coherent six operations.

The objects of Ξ are derived schemes over an implicit base S. The ∞-category of morphisms from X to Y in Ξ will be a subcategory of the ∞-category $\mathbf{Corr}^{\mathcal{L}}((\mathrm{dSch}_{\mathrm{X}\times\mathrm{Y}})_{/\mathrm{K}})^{\mathrm{op}}$ defined in [EHK+19, Appendix B]. A morphism from X to Y is a span

$$\begin{array}{ccccc} & & (\mathrm{Z}, \xi) & & \\ & \swarrow^{f} & & \searrow^{g} & \\ \mathrm{X} & & & & \mathrm{Y} \end{array}$$

where g is locally of finite type and $\xi \in \mathrm{K}(\mathrm{Z})$. One composes such spans by taking pullbacks of schemes and external sums of K-theory elements. A morphism $(\mathrm{Z}, \xi) \to (\mathrm{W}, \eta)$ between such spans is a span from Z to W over $\mathrm{X} \times \mathrm{Y}$:

$$\begin{array}{ccccc} & & (\mathrm{Z}, \xi) & & \\ & \swarrow^{f} & \uparrow{\scriptstyle p} & \searrow^{g} & \\ \mathrm{X} & & \mathrm{T} & & \mathrm{Y} \\ & \nwarrow_{h} & \downarrow{\scriptstyle q} & \nearrow_{k} & \\ & & (\mathrm{W}, \eta) & & \end{array}$$

where p is proper and q is quasi-smooth, together with an isomorphism $p^*(\xi) \simeq q^*(\eta) + \mathcal{L}_q$ in $\mathrm{K}(\mathrm{T})$. The symmetric monoidal structure is given by the product of schemes and the external sum of K-theory elements.

Definition 1. A *coherent formalism of six operations* is a right-lax symmetric monoidal functor

$$\mathrm{D}\colon \Xi \to \mathrm{Pr}^{\mathrm{L}}.$$

Claim 2. *This encodes in particular the following data for $f\colon \mathrm{Y} \to \mathrm{X}$ and $\xi \in \mathrm{K}(\mathrm{X})$:*

(i) *the adjunction (f^*, f_*)*

(ii) *the adjunction $(f_!, f^!)$ for f locally of finite type*

(iii) *the endofunctor Σ^{ξ} of $\mathrm{D}(\mathrm{X})$*

(iv) *the base change isomorphism $g^* f_! \simeq f'_! g'^*$ for any $g\colon \mathrm{X}' \to \mathrm{X}$*

(v) *the isomorphisms $f^*\Sigma^{\xi} \simeq \Sigma^{f^*\xi} f^*$ and $f_!\Sigma^{f^*\xi} \simeq \Sigma^{\xi} f_!$*

(vi) *the natural transformation $\mathfrak{s}_f\colon f_! \to f_*$ for f locally of finite type and separated*

(vii) *the natural transformation $\mathfrak{p}_f\colon \Sigma^{\mathcal{L}_f} f^* \to f^!$ for f quasi-smooth*

(viii) *the fact that $\mathfrak{s}_f$ is an isomorphism when f is proper ($\Leftrightarrow$ f and Δ_f are proper)*

Date: March 19, 2020.

(ix) *the fact that* $\mathfrak{p}_f$ *is an isomorphism when* f *is smooth* ($\Leftrightarrow$ f *and* Δ_f *are quasi-smooth)*

(x) *the fact that* f_* *and* $f_!$ *are fully faithful when* f *is an open immersion*

(xi) *the presentably symmetric monoidal structure on* $\mathrm{D}(\mathrm{X})$

(xii) *the symmetric monoidal structure on* f^*

(xiii) *the* $\mathrm{D}(\mathrm{X})$*-linear structure on* $f_!$

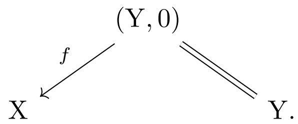
(xiv) *the* $\mathrm{D}(\mathrm{X})$*-linear structure on* Σ^ξ

(xv) *the* $\mathrm{D}(\mathrm{X})$*-linear structure on* $\mathfrak{s}_f \colon f_! \to f_*$

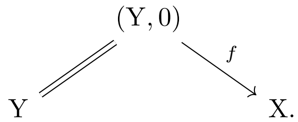
(xvi) *the* $\mathrm{D}(\mathrm{X})$*-linear structure on* $\mathfrak{p}_f \colon \Sigma^{\mathcal{L}_f} f^* \to f^!$

(xvii) *the symmetric monoidal transformation* $\mathrm{K} \to \mathrm{D}$*,* $\xi \mapsto \Sigma^\xi \mathbf{1}$

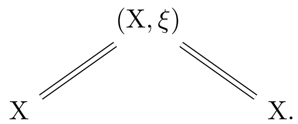
Proof. (i) The functor f^* is the image of the span



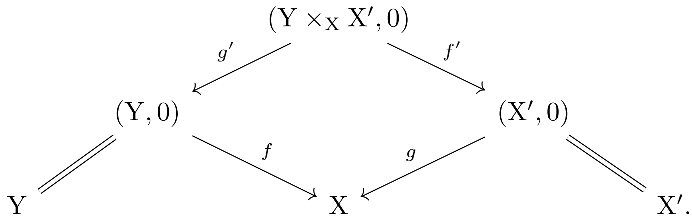
(ii) The functor $f_!$ is the image of the span



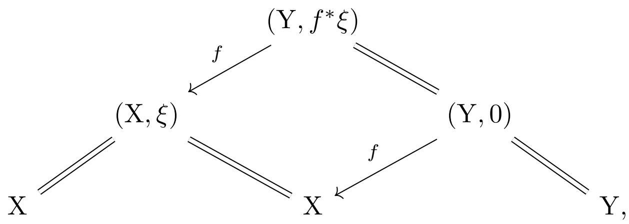
(iii) The functor Σ^ξ is the image of the span



(iv) The base change isomorphism comes from the 2-span

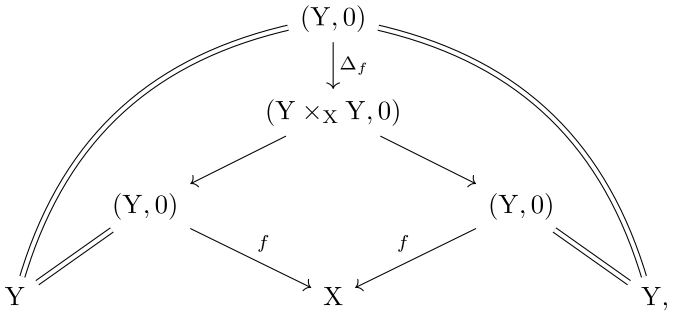


(v) The isomorphism $f^*\Sigma^\xi \simeq \Sigma^{f^*\xi} f^*$ comes from the 2-span



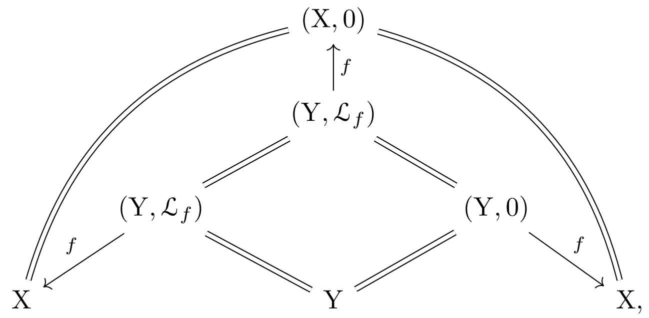
and similarly for $f_!\Sigma^{f^*\xi} \simeq \Sigma^\xi f_!$.

(vi) The adjoint $f^* f_! \to \mathrm{id}$ of $\mathfrak{s}_f$ is the image of the ascending 2-morphism



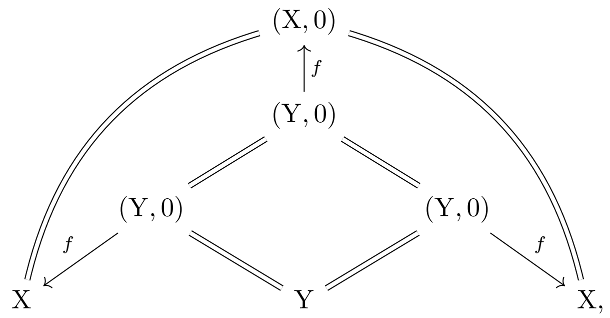
using that Δ_f is proper.

(vii) The adjoint $f_! \Sigma^{\mathcal{L}_f} f^* \to \mathrm{id}$ of $\mathfrak{p}_f$ is the image of the ascending 2-morphism



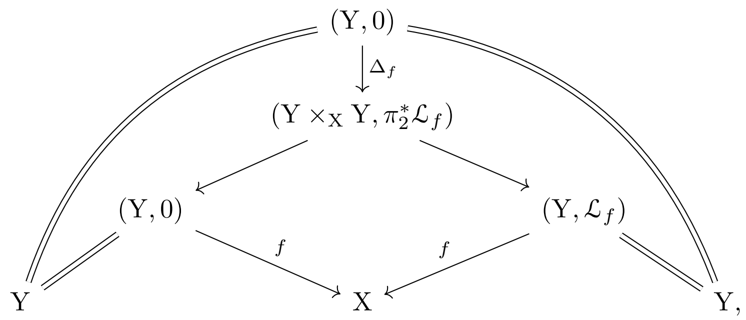
using that f is quasi-smooth.

(viii) There is a transformation $\mathrm{id} \to f_! f^*$ induced by the descending 2-morphism



using that f is proper. This 2-morphism and that of (vi) are the unit and counit of an adjunction in Ξ.

(ix) There is a transformation $\mathrm{id} \to \Sigma^{\mathcal{L}_f} f^* f_!$ induced by the descending 2-morphism



using that Δ_f is quasi-smooth. This 2-morphism and that of (vii) are the unit and counit of an adjunction in Ξ.

(x) This follows from the fact that the 2-morphism from (ix) is an isomorphism when f is an open immersion.

(xi) Every $X \in \Xi$ has a structure of commutative algebra, with multiplication $X^{\otimes I} \to X$ given by

$$\begin{array}{ccc} & (X, 0) & \\ {\scriptstyle \Delta}\swarrow & & \searrow\!\!\!= \\ X^I & & X. \end{array}$$

(xii) $X \xleftarrow{f} (Y, 0) = Y$ has a structure of morphism of commutative algebras in Ξ.

(xiii) $Y = (Y, 0) \xrightarrow{f} X$ has a structure of morphism of X-modules in Ξ.

(xiv) $X = (X, \xi) = X$ has a structure of morphism of X-modules in Ξ.

(xv) The 2-morphism from (vi) has a structure of 2-morphism of X-modules in Ξ.

(xvi) The 2-morphism from (vii) has a structure of 2-morphism of X-modules in Ξ.

(xvii) There is a morphism of commutative monoids $K \to \mathrm{Maps}_{\Xi}(\mathbf{1}, -)$ sending $\xi \in K(X)$ to

$$\begin{array}{ccc} & (X, \xi) & \\ \swarrow & & \searrow\!\!\!= \\ S & & X. \end{array}$$

$\square$

References

[EHK+19] E. Elmanto, M. Hoyois, A. A. Khan, V. Sosnilo, and M. Yakerson, *Modules over algebraic cobordism*, 2019, arXiv:1908.02162v1