Milnor excision for motivic spectra

Marc Hoyois

(joint work with Elden Elmanto, Ryomei Iwasa, Shane Kelly)

The title of the talk refers to the following theorem, where $\mathrm{SH}(A)$ denotes the Morel–Voevodsky stable ∞-category of motivic spectra over A:

Main Theorem ([4]). *Let $A \to B$ be a morphism of commutative rings mapping an ideal $I \subset A$ isomorphically onto an ideal $J \subset B$. Then the following square of ∞-categories is cartesian:*

$$\begin{array}{ccc} \mathrm{SH}(A) & \longrightarrow & \mathrm{SH}(B) \\ \downarrow & & \downarrow \\ \mathrm{SH}(A/I) & \longrightarrow & \mathrm{SH}(B/J). \end{array}$$

In particular, every motivic spectrum $E \in \mathrm{SH}(A)$ gives rise to a cartesian square of spectra

$$\begin{array}{ccc} R\Gamma(\operatorname{Spec} A, E) & \longrightarrow & R\Gamma(\operatorname{Spec} B, E) \\ \downarrow & & \downarrow \\ R\Gamma(\operatorname{Spec} A/I, E) & \longrightarrow & R\Gamma(\operatorname{Spec} B/J, E). \end{array}$$

Examples of such morphisms $(A, I) \to (B, J)$ include the coordinate axes

$$(k[x, y]/(xy), (x)) \to (k[x], (x)),$$

the "Rim square"

$$(\mathbb{Z}[x]/(x^p - 1), (x - 1)) \to (\mathbb{Z}[\zeta_p], (\zeta_p - 1)),$$

and the desingularization of an affine curve. Another key example, first considered by Huber and Kelly [5], is the localization map $(V, \mathfrak{p}) \to (V_\mathfrak{p}, \mathfrak{p}V_\mathfrak{p})$ for $\mathfrak{p}$ a prime ideal in a valuation ring V.

Examples of cohomology theories $R\Gamma(\,\cdot\,, E)$ defined on all schemes include:

- $\mathbb{A}^1$-invariant algebraic K-theory $\mathrm{KH} = L_{\mathbb{A}^1}\mathbb{K}$, where $\mathbb{K}$ denotes localizing (i.e., nonconnective) K-theory (Weibel).
- $\mathbb{A}^1$-invariant motivic cohomology (Spitzweck), which is known to agree with the cdh motivic cohomology of Elmanto–Morrow in equicharacteristic (and is expected to in general).
- $\mathbb{A}^1$-invariant symmetric Grothendick–Witt theory $L_{\mathbb{A}^1}\mathbb{GW}^{\mathrm{s}}$, where $\mathbb{GW}^{\mathrm{s}}$ denotes the localizing Grotendieck–Witt theory of homotopy-symmetric forms (Calmès–Harpaz–Nardin).
- Étale cohomology with coefficients in $\mathcal{F}$, where $\mathcal{F}$ is a torsion étale sheaf of abelian groups over a scheme S, whose torsion is coprime to the residual characteristics of S.

The main theorem follows from Theorems A and B below using ideas of Bhatt and Mathew [1].

Theorem A ([4])**.** *Let V be a valuation ring and $\mathfrak{p} \subset V$ a prime ideal. Then the following square of ∞-categories is cartesian:*

$$\begin{array}{ccc} \mathrm{SH}(V) & \longrightarrow & \mathrm{SH}(V_{\mathfrak{p}}) \\ \downarrow & & \downarrow \\ \mathrm{SH}(V/\mathfrak{p}) & \longrightarrow & \mathrm{SH}(\kappa(\mathfrak{p})). \end{array}$$

Using the localization property and the finitary nature of SH, Theorem A is reduced to the following statement: if V has finite rank and

$$\operatorname{Spec} V_{\mathfrak{p}} - \{\mathfrak{p}\} \xrightarrow{h} \operatorname{Spec} V_{\mathfrak{p}} \xrightarrow{j} \operatorname{Spec} V$$

are the canonical open immersions, then the natural transformation

$$j_! h_! \to j_* h_! \colon \mathrm{SH}(\operatorname{Spec} V_{\mathfrak{p}} - \{\mathfrak{p}\}) \to \mathrm{SH}(\operatorname{Spec} V)$$

is an isomorphism. Using the cdh descent of motivic spectra proved by Cisinski, this is in turn implied by the same statement for $\mathrm{SH}_{\mathrm{cdh}}$, which is the analogue of SH defined using the cdh site instead of the smooth Nisnevich site. The point is that pushforwards along quasi-compact open immersions are compatible with the cdh topology, allowing us to further reduce the statement to the level of presheaves. There it becomes an easy direct computation in light of the following fact, which uses that V is a valuation ring in an essential way: if X is a connected scheme, the image of any map $X \to \operatorname{Spec} V$ is an interval in the specialization poset.

Theorem B ([3])**.** *Let X be a qcqs scheme. Then the homotopy dimension of the cdh ∞-topos of X is at most the valuative dimension of X.*

Recall that an ∞-topos has *homotopy dimension* $\leq d$ if every d-connective sheaf admits a global section. This implies in particular that abelian cohomology vanishes in degrees $> d$ (since cohomology classes of degree $n+1$ classify n-gerbes, which are n-connective). The *valuative dimension* of a scheme is a variation of the Krull dimension introduced by Jaffard, which is surprisingly well-behaved for non-noetherian schemes. For an integral scheme X with fraction field K, $\mathrm{vdim}(X)$ is the supremum of the lengths of chains of valuation subrings of K centered on X, and one can extend this notion to arbitrary schemes by taking the supremum over all irreducible components. The valuative dimension agrees with the Krull dimension for locally noetherian schemes as well as for valuation rings, but it has the advantage of being a birational invariant in general.

The proof of Theorem B uses the Riemann–Zariski space $\mathrm{RZ}(X)$ of an integral scheme X, which is the limit of all blow-ups of X (one can first reduce to the integral case using the fact that the space of generic points of a qcqs scheme is totally separated). The canonical map $p\colon \mathrm{RZ}(X) \to X$ induces a geometric morphism of ∞-topoi

$$p^* \colon \mathrm{Shv}_{\mathrm{cdh}}(\mathrm{Sch}_X^{\mathrm{fp}}) \to \mathrm{Shv}_{\mathrm{Nis}}(\mathrm{RZ}(X)).$$

The valuative dimension of X turns out to be the Krull dimension of $\mathrm{RZ}(X)$, which by a theorem of Clausen and Mathew is an upper bound for the homotopy dimension of $\mathrm{Shv}_{\mathrm{Nis}}(\mathrm{RZ}(X))$ [2]. Hence any $\mathrm{vdim}(X)$-connective cdh sheaf on X has a section over some blow-up of X, which can be descended to a section over X using cdh descent and the induction hypothesis.

References

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