1 Preliminaries on the stack $\overline{\mathcal{M}}_{0,n}$

Let k be an algebraically closed field and $n \geq 3$. An *n-pointed stable rational curve* over k is a reduced and connected curve C over k with n distinct smooth k-points $x_1, \ldots, x_n$ such that

1. the singular points of C are ordinary double points;
2. each irreducible component of C is isomorphic to $\mathbb{P}^1_k$;
3. on each irreducible component, the number of marked points plus the number of singular points is at least 3;
4. C has genus 0, i.e., $H^1(C, \mathcal{O}_C) = 0$.

A morphism $(C, x_1, \ldots, x_n) \to (C', x'_1, \ldots, x'_n)$ between such curves must send x_i to x'_i. The conditions guarantee that such a curve has no nontrivial automorphism. Indeed, condition 4 imply that C is a "tree", and since C is of finite type any "maximal path" contains marked points in its extremal components. Thus if f is an automorphism of C, it must map any such path, and therefore any component within it, to itself, otherwise there would be a cycle. This shows that a singular point is mapped to itself, and conditions 2 and 3 imply that f is the identity.

If S is a scheme over $\mathbb{C}$, let $\overline{\mathcal{M}}_{0,n}(S)$ be the groupoid of n-pointed stable rational curves over S. An object in this groupoid is a flat and proper morphism of $\mathbb{C}$-schemes $\pi\colon C \to S$ with n sections such that every geometric fiber in an n-pointed stable rational curve. This groupoid has no π_1 and we shall thus view $\overline{\mathcal{M}}_{0,n}(S)$ as the set of its isomorphism classes. Because pullback preserves flat and proper morphisms as well as fibers, $\overline{\mathcal{M}}_{0,n}$ is a contravariant functor of S.

We denote by $\partial\overline{\mathcal{M}}_{0,n}$ the subfunctor consisting of curves all of whose geometric fibers are singular. For every partition $K = (K_1, K_2)$ of $\{1, \ldots, n\}$ with $|K_i| = n_i \geq 2$, we denote by $\partial_K\overline{\mathcal{M}}_{0,n}$ the subfunctor of $\partial\overline{\mathcal{M}}_{0,n}$ consisting of curves whose geometric fibers are of the form $C_1 \vee C_2$ with $x_j \in C_1$ if $j \in K_1$ and $x_j \in C_2$ if $j \in K_2$. There is a natural bijection

$$\overline{\mathcal{M}}_{0,n_1+1} \times \overline{\mathcal{M}}_{0,n_2+1} \to \partial_K\overline{\mathcal{M}}_{0,n}$$

which identifies the last section of the first curve to the first section of the second curve and reorders the remaining sections appropriately.

In [Knu83a, Knu83b] Knudsen proves the following:

Theorem 1.1. *The functor $\overline{\mathcal{M}}_{0,n}\colon (\mathcal{S}\mathrm{ch}/\mathbb{C})^\circ \to \mathcal{S}\mathrm{et}$ is represented by a smooth projective variety of dimension $n-3$. The functor $\partial\overline{\mathcal{M}}_{0,n}$ is a closed subvariety of codimension 1 whose irreducible components are the subvarieties $\partial_K\overline{\mathcal{M}}_{0,n}$.*

He also proves that there is a contraction morphism $\overline{\mathcal{M}}_{0,n+1} \to \overline{\mathcal{M}}_{0,n}$ that forgets, say, the last section in a way that preserves stability. More precisely, the image of $\pi\colon C \to S$ is a curve $\pi'\colon C' \to S$ such that there exists a morphism $f\colon C \to C'$, commuting with structure maps and sections, which on each geometric fiber is either an isomorphism or contracts the irreducible component of x_{n+1} to a closed point (in case condition 3 would otherwise be violated). It is then clear that the closed immersion $\phi_K\colon \partial_K\overline{\mathcal{M}}_{0,n} \to \overline{\mathcal{M}}_{0,n}$ is right inverse to any morphism $\overline{\mathcal{M}}_{0,n} \to \overline{\mathcal{M}}_{0,n_1+1} \times \overline{\mathcal{M}}_{0,n_2+1}$ that on the first factor (resp. the second factor) contracts consecutively all sections indexed by K_2 (resp. by K_1) except one. Thus, the map ϕ_K^* induced by ϕ_K on cohomology is surjective, while the map ϕ_{K*} on homology is injective.

The following description of $H^*(\overline{\mathcal{M}}_{0,n}, \mathbb{Z})$ is from [Kee92].

Theorem 1.2. *Let $n \geq 3$. The cycle class map $A^*(\overline{\mathcal{M}}_{0,n}) \to H^*(\overline{\mathcal{M}}_{0,n}, \mathbb{Z})$ is an isomorphism. Moreover, the ring $A^*(\overline{\mathcal{M}}_{0,n})$ is generated by the cycles ϕ_K, where $K = (K_1, K_2)$ is a partition of $\{1, \ldots, n\}$ with $|K_i| \geq 2$, with only the following relations:*

1. $\phi_{(K_1,K_2)} = \phi_{(K_2,K_1)}$;

2. for any pairwise distinct indices i, j, k, l in $\{1, \dots, n\}$,

$$\sum_{\substack{i,j\in K_1\\k,l\in K_2}} \phi_K = \sum_{\substack{i,k\in K_1\\j,l\in K_2}} \phi_K = \sum_{\substack{i,l\in K_1\\j,k\in K_2}} \phi_K; \tag{1}$$

3. $\phi_K\phi_L = 0$ unless $K_i \subseteq L_j$ for some $i, j \in \{1, 2\}$.

We denote by $\delta_K \in H^2(\overline{\mathcal{M}}_{0,n}, \mathbb{Z})$ the image of the prime divisor ϕ_K by the cycle class map.

2 Axioms for genus 0 Gromov–Witten invariants

Let V be a smooth projective variety over $\mathbb{C}$ of complex dimension d. A *tree level system of Gromov–Witten classes* for V is a family of $\mathbb{Q}$-linear maps

$$I^V_{0,n,\beta} \colon H^*(V, \mathbb{Q})^{\otimes n} \to H^*(\overline{\mathcal{M}}_{0,n}, \mathbb{Q}),$$

defined for all $n \geq 3$ and all homology classes $\beta \in H_2(V, \mathbb{Z})$, required to satisfy the following axioms. (From now on, cohomology is understood to have rational coefficients.)

Effectivity axiom. $I^V_{0,n,\beta} = 0$ unless β is is effective, i.e., has nonnegative intersection number with the image of any ample line bundle in $H^2(V, \mathbb{Z})$, or equivalently, has nonnegative degree in any projective embedding.

Σ_n-invariance axiom. $I^V_{0,n,\beta}$ is Σ_n-equivariant (Σ_n acts on $H^*(V, \mathbb{Q})^{\otimes n}$ with the Koszul sign convention and on $\overline{\mathcal{M}}_{0,n}$ by permuting marked points).

Dimension axiom. $I^V_{0,n,\beta}$ is a graded map of degree $2\langle K_V, \beta\rangle - 2d$, where $K_V = c_1(\Omega_V^{\wedge d}) \in H^2(V, \mathbb{Z})$ is the canonical class of V.

Identity axiom. If $n \geq 4$,

$$I^V_{0,n,\beta}(\gamma_1 \otimes \cdots \otimes \gamma_{n-1} \otimes e) = \pi_n^* I^V_{0,n-1,\beta}(\gamma_1 \otimes \cdots \otimes \gamma_{n-1}),$$

where $\pi_n \colon \overline{\mathcal{M}}_{0,n} \to \overline{\mathcal{M}}_{0,n-1}$ forgets the nth marked point.

For $n = 3$ and $\beta \neq 0$,

$$I^V_{0,3,\beta}(\gamma_1 \otimes \gamma_2 \otimes e) = 0.$$

Divisor axiom. If $n \geq 4$ and $\delta \in H^2(V, \mathbb{Q})$,

$$\pi_{n*} I^V_{0,n,\beta}(\gamma_1 \otimes \cdots \otimes \gamma_{n-1} \otimes \delta) = \langle \delta, \beta\rangle I^V_{0,n-1,\beta}(\gamma_1 \otimes \cdots \otimes \gamma_{n-1}).$$

Mapping to point axiom.

$$I^V_{0,n,0}(\gamma_1 \otimes \cdots \otimes \gamma_n) = \begin{cases} \left(\int_V \gamma_1 \wedge \cdots \wedge \gamma_n\right) e & \text{if } \sum_{i=1}^n |\gamma_i| = 2d, \\ 0 & \text{otherwise.} \end{cases}$$

Splitting axiom. Let $K = (K_1, K_2)$ be a partition of $\{1, \dots, n\}$, with $|K_i| = n_i \geq 2$. Then

$$\phi_K^* I^V_{0,n,\beta}(\gamma_1 \otimes \cdots \otimes \gamma_n) = \pm \sum_{\beta=\beta_1+\beta_2} (I^V_{0,n_1+1,\beta_1} \otimes I^V_{0,n_2+1,\beta_2})((\otimes_{j\in K_1}\gamma_j) \otimes \Delta \otimes (\otimes_{j\in K_2}\gamma_j)),$$

where $\Delta \in H^*(V, \mathbb{Q})^{\otimes 2}$ is the image of the diagonal cycle and the sign is the Koszul sign arising from moving the K_1- and K_2-indexed classes past one another. The effectivity axiom guarantees that the sum on the right-hand side is essentially finite.

Motivic axiom. The map $I^V_{0,n,\beta}$ comes from a morphism of Chow motives $V^n \to \overline{\mathcal{M}}_{0,n}(2d - 2\langle K_V, \beta\rangle)$. This axiom is not used in the next section.

3 The first reconstruction theorem

The results of this section are from [KM94].

The *codimension* of a nonzero homogeneous class $\alpha \in H^*(\overline{\mathcal{M}}_{0,n})$ is $2n-6-|\alpha|$, where $\alpha \in H^{|\alpha|}(\overline{\mathcal{M}}_{0,n})$.

Proposition 3.1. *Any tree level system of GW classes is determined by its classes of codimension zero. In fact, a homogeneous class $I^V_{0,n,\beta}(\gamma_1 \otimes \cdots \otimes \gamma_n)$ of positive codimension is determined by codimension zero classes with at most $n-1$ arguments.*

Proof. Assume classes of codimension zero are known. We compute an arbitrary homogeneous class $\alpha = I^V_{0,n,\beta}(\gamma_1 \otimes \cdots \otimes \gamma_n)$ by induction on $n \geq 3$. If $n = 3$, since $\overline{\mathcal{M}}_{0,3}$ is zero-dimensional, the class is either zero or of codimension zero.

Let $n \geq 4$. For every partition $K = (K_1, K_2)$ of $\{1, \dots, n\}$ with $|K_i| = n_i \geq 2$, the splitting axiom expresses $\phi_K^*\alpha$ in terms of GW classes with a strictly smaller n (since $n_i + 1 < n$), so it is known by induction hypothesis. Consider the exact sequence

$$0 \to \bigcap_K \ker(\phi_K^*) \to H^*(\overline{\mathcal{M}}_{0,n}) \to \bigoplus_K H^*(\overline{\mathcal{M}}_{0,n_1+1}) \otimes_{\mathbb{Q}} H^*(\overline{\mathcal{M}}_{0,n_2+1}),$$

in which the last arrow is sum of the ϕ_K^*, and suppose that we can prove

$$\bigcap_K \ker(\phi_K^*) = H^{2n-6}(\overline{\mathcal{M}}_{0,n}). \tag{2}$$

Then, since α is homogeneous, it either has codimension zero or is determined by all the $\phi_K^*\alpha$.

The inclusion from right to left in (2) is clear because the dimension of $\overline{\mathcal{M}}_{0,n_1+1} \times \overline{\mathcal{M}}_{0,n_2+1}$ is one less than the dimension of $\overline{\mathcal{M}}_{0,n}$. Conversely, consider a homogeneous class $\alpha \in H^i(\overline{\mathcal{M}}_{0,n})$ with $\phi_K^*\alpha = 0$ for all K. Thus $\alpha \wedge \delta_K = \phi_{K*}\phi_K^*\alpha = 0$ for all K, and so $\alpha \wedge \eta = 0$ for every homogeneous class η with $|\eta| > 0$ (any such class being a rational polynomial in the δ_K's with no constant term, by Theorem 1.2). Since the pairings $H^i(\overline{\mathcal{M}}_{0,n}) \otimes_{\mathbb{Q}} H^{2n-6-i}(\overline{\mathcal{M}}_{0,n}) \to \mathbb{Q}$, $\alpha \otimes \eta \mapsto \int \alpha \wedge \eta$, are perfect, either $\alpha = 0$ or $i = 2n-6$. ☐

Theorem 3.2 (First Reconstruction Theorem)**.** *Suppose that $H^*(V)$ is generated as a ring by $H^2(V)$. Then any tree level system of GW classes is determined by its classes of the form $I^V_{0,3,\beta}(\gamma_1 \otimes \gamma_2 \otimes \gamma_3)$ with $\min_i|\gamma_i| = 2$.*

Let $B \subset H_2(V, \mathbb{Z})$ denote the monoid of effective classes. We endow the set of triples (n, β, k) with $n \geq 3$, $\beta \in B$, and $k \geq 0$ with the lexicographic order: $(n, \beta, k) \leq (n', \beta', k')$ if $n < n'$, or if $n = n'$ and $\beta < \beta'$ (which means that $\beta = \beta' + \beta''$ for some $\beta'' \in B - \{0\}$), or if $n = n'$, $\beta = \beta'$, and $k \leq k'$. The monoid B is obviously torsion-free and it is finitely generated because $B \otimes_{\mathbb{Z}} \mathbb{Q}$ is a cone in the finite-dimensional vector space $H_2(V, \mathbb{Q})$. It follows that this partial order is well-founded, and we are therefore entitled to reason by induction with respect to it.

In addition to the classes in the statement of the theorem, we can therefore assume that the classes $I^V_{0,n',\beta'}(\gamma'_1 \otimes \cdots \otimes \gamma'_{n'})$ are known for all $(n', \beta', \min_i|\gamma'_i|) < (n, \beta, \min_i|\gamma_i|)$, and we must show that $I^V_{0,n,\beta}(\gamma_1 \otimes \cdots \otimes \gamma_n)$ is uniquely determined by the axioms. By Proposition 3.1, we are already done unless $I^V_{0,n,\beta}(\gamma_1 \otimes \cdots \otimes \gamma_n)$ has codimension zero, which we assume from now on. The point of this assumption is that the map $\pi_{n*}\colon H^{2n-6}(\overline{\mathcal{M}}_{0,n}) \to H^{2(n-1)-6}(\overline{\mathcal{M}}_{0,n-1})$ appearing in the divisor axiom is an isomorphism.

If $\beta = 0$, the class is determined by the mapping to point axiom. If $n \geq 4$ and $\min_i|\gamma_i|$ is 0 (resp. 2), it is determined by the identity axiom (resp. the divisor axiom and the fact that it has codimension zero), Σ_n-invariance, and the induction hypothesis. If $n = 3$ and $\min_i|\gamma_i|$ is 0 (resp. 2), it is determined by the identity axiom and Σ_n-invariance (resp. by assumption).

Assume therefore that $\beta > 0$ and $\min_i|\gamma_i| > 2$. By reordering the arguments, we can assume that $|\gamma_n| = \min_i|\gamma_i|$. Since $H^*(V)$ is generated by $H^2(V)$, we can further assume that $\gamma_n = \delta \wedge \epsilon$ where $|\delta| = 2$ and $|\epsilon| \geq 2$.

Lemma 3.3. *Let i,j,k,l be four distinct indices in $\{1,\dots,n\}$, $n \geq 4$, and let $I^V_{0,n,\beta}(\gamma_1 \otimes \cdots \otimes \gamma_n)$ be a GW class. Then for appropriate signs, the class*

$$\pm \phi_{i,j*}(e \otimes I^V_{0,n-1,\beta}(\gamma_i \wedge \gamma_j \otimes (\otimes_{t\neq i,j}\gamma_t))) \pm \phi_{k,l*}(e \otimes I^V_{0,n-1,\beta}(\gamma_k \wedge \gamma_l \otimes (\otimes_{t\neq k,l}\gamma_t)))$$
$$\pm \phi_{i,k*}(e \otimes I^V_{0,n-1,\beta}(\gamma_i \wedge \gamma_k \otimes (\otimes_{t\neq i,k}\gamma_t))) \pm \phi_{j,l*}(e \otimes I^V_{0,n-1,\beta}(\gamma_j \wedge \gamma_l \otimes (\otimes_{t\neq j,l}\gamma_t)))$$

is determined by GW classes of lower order. Here $\phi_{i,j} \colon \overline{\mathcal{M}}_{0,3} \times \overline{\mathcal{M}}_{0,n-1} \to \overline{\mathcal{M}}_{0,n}$ is the divisor for the partition (K_1, K_2) with $K_1 = \{i,j\}$, and $e \in H^*(\overline{\mathcal{M}}_{0,3}, \mathbb{Q})$ is the ring unit.

Proof. We use the relations (1):

$$\sum_{\substack{i,j\in K_1\\ k,l\in K_2}} \phi_{K*}\phi_K^* I^V_{0,n,\beta}(\gamma_1 \otimes \cdots \otimes \gamma_n) = \sum_{\substack{i,k\in K_1\\ j,l\in K_2}} \phi_{K*}\phi_K^* I^V_{0,n,\beta}(\gamma_1 \otimes \cdots \otimes \gamma_n).$$

By the splitting axiom, this becomes

$$\sum_{\substack{i,j\in K_1\\ k,l\in K_2}} \sum_{\beta=\beta_1+\beta_2} \pm\phi_{K*}(I^V_{0,n_1+1,\beta_1} \otimes I^V_{0,n_2+1,\beta_2})((\otimes_{m\in K_1}\gamma_m) \otimes \Delta \otimes (\otimes_{m\in K_2}\gamma_m))$$
$$= \sum_{\substack{i,k\in K_1\\ j,l\in K_2}} \sum_{\beta=\beta_1+\beta_2} \pm\phi_{K*}(I^V_{0,n_1+1,\beta_1} \otimes I^V_{0,n_2+1,\beta_2})((\otimes_{m\in K_1}\gamma_m) \otimes \Delta \otimes (\otimes_{m\in K_2}\gamma_m)).$$

All nonzero terms only involve classes of order strictly less than $(n-1,\beta)$, except those terms in which $n_1 = 2$ and $\beta_1 = 0$, or $n_2 = 2$ and $\beta_2 = 0$. There are two such terms on each side of the equality. Let's look at the term on the left-hand side indexed by $K_1 = \{i,j\}$:

$$\pm\phi_{i,j*}(I^V_{0,3,0} \otimes I^V_{0,n-1,\beta})(\gamma_i \otimes \gamma_j \otimes \Delta \otimes (\otimes_{m\neq i,j}\gamma_m))$$

(the order of γ_i and γ_j only affects the sign, as the following computation shows). If we write $\Delta = \sum_a \alpha_a \otimes \beta_a$, then $I^V_{0,3,0}(\gamma_i \otimes \gamma_j \otimes \alpha_a) = (\int_V \gamma_i \wedge \gamma_j \wedge \alpha_a)e$ by the mapping to point axiom (if the integrand is not a top class, let the integral be zero). Poincaré duality and the computation

$$\sum_a \int_V \left(\int_V \gamma \wedge \alpha_a\right) \beta_a \wedge \gamma' = \int_{V\times V} p_1^*(\gamma) \wedge p_2^*(\gamma') \wedge \Delta = \int_{\Delta(V)} p_1^*(\gamma) \wedge p_2^*(\gamma') = \int_V \gamma \wedge \gamma'$$

show that for any cohomology class γ, $\sum_a (\int_V \gamma \wedge \alpha_a)\beta_a = \gamma$. Thus,

$$\pm(I^V_{0,3,0} \otimes I^V_{0,n-1,\beta})(\gamma_i \otimes \gamma_j \otimes \Delta \otimes (\otimes_{m\neq i,j}\gamma_m)) = \pm e \otimes I^V_{0,n-1,\beta}(\gamma_i \wedge \gamma_j \otimes (\otimes_{m\neq i,j}\gamma_m)).$$

A similar computation puts the other three terms in the desired form. $\square$

The lemma applied to the indices $1, 2, n, n+1 \in \{1,\dots,n+1\}$ and the induction hypothesis show that the expression

$$\pm (\phi_{1,2})_*(e \otimes I^V_{0,n,\beta}(\gamma_1 \wedge \gamma_2 \otimes \gamma_3 \otimes \cdots \otimes \gamma_{n-1} \otimes \delta \otimes \epsilon))$$
$$\pm (\phi_{n,n+1})_*(e \otimes I^V_{0,n,\beta}(\delta \wedge \epsilon \otimes \gamma_1 \otimes \cdots \otimes \gamma_{n-1}))$$
$$\pm (\phi_{1,n})_*(e \otimes I^V_{0,n,\beta}(\gamma_1 \wedge \delta \otimes \gamma_2 \otimes \cdots \otimes \gamma_{n-1} \otimes \epsilon))$$
$$\pm (\phi_{2,n+1})_*(e \otimes I^V_{0,n,\beta}(\gamma_2 \wedge \epsilon \otimes \gamma_1 \otimes \gamma_3 \otimes \cdots \otimes \gamma_{n-1} \otimes \delta))$$

is uniquely determined. Since ϕ_{K*} is injective, the second term determines $I^V_{0,n,\beta}(\gamma_1 \otimes \cdots \otimes \gamma_n)$. But the other three terms are also determined by induction hypothesis, since $|\delta| < \min_i |\gamma_i|$ and $|\epsilon| < \min_i |\gamma_i|$. This completes the proof of Theorem 3.2.

Further restrictions on the classes of Theorem 3.2 are obtained by noting that the class $I^V_{0,3,\beta}(\gamma_1 \otimes \gamma_2 \otimes \gamma_3)$ can be nonzero only if

$$|\gamma_1| + |\gamma_2| + |\gamma_3| = 2d - 2\langle K_V, \beta\rangle,$$

by the dimension axiom. Since $|\gamma_i| \leq 2d$ and $\min_i |\gamma_i| = 2$, it will therefore vanish unless β satisfies

$$-d - 1 \leq \langle K_V, \beta\rangle \leq d - 3.$$

If (and only if) $\ker\langle K_V, -\rangle \cap B = 0$, there are only finitely many such effective β.

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