CDH DESCENT IN EQUIVARIANT HOMOTOPY K-THEORY

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Abstract. We construct geometric models for classifying spaces of linear algebraic groups in G-equivariant motivic homotopy theory, where G is a tame group scheme. As a consequence, we show that the equivariant motivic spectrum representing the homotopy K-theory of G-schemes (which we construct as an E_∞-ring) is stable under arbitrary base change, and we deduce that the homotopy K-theory of G-schemes satisfies cdh descent.

Contents

1. Introduction

Let $K(X)$ and $K^B(X)$ denote the connective and nonconnective K-theory spectra of a quasi-compact quasi-separated scheme X [TT90]. The homotopy K-theory spectrum $KH(X)$ was introduced by Weibel in [Wei89]: it is the geometric realization of the simplicial spectrum $K^B(\Delta^\bullet \times X)$, where

$$\Delta^n = \operatorname{Spec} \mathbb{Z}[t_0, \ldots, t_n]/(\textstyle\sum_i t_i - 1)$$

is the standard algebraic n-simplex. There are natural transformations $K \to K^B \to KH$, which are equivalences on regular schemes.

Haesemeyer [Hae04] (in characteristic zero) and Cisinski [Cis13] (in general) proved that homotopy K-theory satisfies descent for Voevodsky's cdh topology. This was a key ingredient in the proof of Weibel's vanishing conjecture for negative K-theory, established in characteristic zero by Cortiñas, Haesemeyer, Schlichting, and Weibel [CHSW08], and up to p-torsion in characteristic $p > 0$ by Kelly [Kel14] (with a simplified proof by Kerz and Strunk [KS17]). More recently, Kerz, Strunk, and Tamme proved that K-theory satisfies "pro-cdh descent" and deduced Weibel's conjecture in complete generality [KST18].

The goal of this paper is to extend the cdh descent result of Cisinski to a suitable class of Artin stacks, namely, quotients of schemes by linearizable actions of linearly reductive algebraic groups. We will introduce a reasonable definition of the homotopy K-theory spectrum $KH(\mathfrak{X})$ for such a stack $\mathfrak{X}$, which agrees with $K(\mathfrak{X})$ when $\mathfrak{X}$ is regular. The "obvious" extension of Weibel's definition works well for quotients by finite or diagonalizable groups, but, for reasons we will explain below, a more complicated definition is preferred in general. Our main results are summarized in Theorem 1.3 below. In a sequel to this paper, joint with Amalendu Krishna, we use these results to prove vanishing theorems for the negative K-theory of tame Artin stacks [HK19].

Let us first introduce some terminology. A morphism of stacks $\mathfrak{Y} \to \mathfrak{X}$ will be called *quasi-projective* if there exists a finitely generated quasi-coherent module $\mathcal{E}$ over $\mathfrak{X}$ and a quasi-compact immersion $\mathfrak{Y} \hookrightarrow \mathbb{P}(\mathcal{E})$ over $\mathfrak{X}$. We say that a stack $\mathfrak{X}$ has the *resolution property* if every finitely generated quasi-coherent module over $\mathfrak{X}$ is the quotient of a locally free module of finite rank. Throughout this paper, we will work over a

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fixed quasi-compact separated (qcs) base scheme B, and we will say that a morphism of B-stacks $\mathfrak{Y} \to \mathfrak{X}$ is *N-quasi-projective* if it is quasi-projective Nisnevich-locally on B. We refer to [Hoy17, §2.7] for the precise definition of a *tame* group scheme over B. The main examples of interest are:

- finite locally free groups of order invertible on B;
- groups of multiplicative type;
- reductive groups, if B has characteristic 0 (i.e., there exists $B \to \operatorname{Spec} \mathbb{Q}$).

Let tqStk_B denote the 2-category of finitely presented B-stacks that have the resolution property, that are global quotient stacks $[X/G]$ for some tame affine group scheme G, and such that the resulting map $[X/G] \to \mathbf{B}G$ is N-quasi-projective.[1] For $\mathfrak{X} \in \mathrm{tqStk}_B$, we let $\mathrm{Sch}_{\mathfrak{X}} \subset (\mathrm{tqStk}_B)_{/\mathfrak{X}}$ be the full subcategory of N-quasi-projective $\mathfrak{X}$-stacks. The Nisnevich (resp. cdh) topology on $\mathrm{Sch}_{\mathfrak{X}}$ is as usual the Grothendieck topology generated by Nisnevich squares (resp. Nisnevich squares and abstract blowup squares). The Nisnevich and cdh topologies on tqStk_B are generated by the corresponding topologies on the slices $\mathrm{Sch}_{\mathfrak{X}}$.

Remark 1.1. If B has characteristic zero, the 2-category tqStk_B includes all Artin stacks of finite presentation, with affine stabilizers, and satisfying the resolution property. Indeed, by a theorem of Gross [Gro17, Theorem 1.1], such stacks have the form $[X/\mathrm{GL}_n]$, where X is a quasi-affine GL_n-scheme.

Remark 1.2. The stacks in tqStk_B share many features with the "tame Artin stacks" considered in [AOV08]. There are two essential differences: our stacks are not required to have finite diagonal, but theirs are not required to have the resolution property.

Theorem 1.3. *Let B be a quasi-compact separated base scheme. There exists a cdh sheaf of E_∞-ring spectra*

$$KH\colon \mathrm{tqStk}_B^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Sp})$$

and an E_∞-map $K \to KH$ with the following properties.

1. *If $\mathfrak{X} \in \mathrm{tqStk}_B$ is regular, the map $K(\mathfrak{X}) \to KH(\mathfrak{X})$ is an equivalence.*
2. *KH is homotopy invariant in the following strong sense: if $p\colon \mathfrak{Y} \to \mathfrak{X}$ is an fpqc torsor under a vector bundle, then $p^*\colon KH(\mathfrak{X}) \to KH(\mathfrak{Y})$ is an equivalence.*
3. *KH satisfies Bott periodicity: for every vector bundle $\mathfrak{V}$ over $\mathfrak{X}$, there is a canonical equivalence of $KH(\mathfrak{X})$-modules $KH(\mathfrak{V} \text{ on } \mathfrak{X}) \simeq KH(\mathfrak{X})$.*
4. *Suppose that $\mathfrak{X} \in \mathrm{Sch}_{\mathbf{B}G}$ where G is an extension of a finite group scheme by a Nisnevich-locally diagonalizable group scheme. Then $KH(\mathfrak{X})$ is the geometric realization of the simplicial spectrum $K^B(\Delta^\bullet \times \mathfrak{X})$.*

From property (1) and the hypercompleteness of the cdh topology, we immediately deduce:

Corollary 1.4. *Suppose that B is noetherian of finite Krull dimension and that every stack in tqStk_B admits a cdh cover by regular stacks, e.g., B is essentially of finite type over a field of characteristic zero. Then the canonical map $K \to KH$ exhibits KH as the cdh sheafification of K.*

The fact that KH is a cdh sheaf means that it is a Nisnevich sheaf and that, for every cartesian square

$$\begin{tikzcd} \mathfrak{W} \arrow[r, hook] \arrow[d] & \mathfrak{Y} \arrow[d, "p"] \\ \mathfrak{Z} \arrow[r, hook, "i"'] & \mathfrak{X} \end{tikzcd}$$

in tqStk_B such that i is a closed immersion, p is N-projective, and p induces an isomorphism $\mathfrak{Y} \smallsetminus \mathfrak{W} \simeq \mathfrak{X} \smallsetminus \mathfrak{Z}$, the induced square of spectra

$$\begin{tikzcd} KH(\mathfrak{X}) \arrow[r] \arrow[d] & KH(\mathfrak{Z}) \arrow[d] \\ KH(\mathfrak{Y}) \arrow[r] & KH(\mathfrak{W}) \end{tikzcd}$$

[1] Quasi-projective $\mathbf{B}G$-stacks almost have the resolution property: they admit a schematic Nisnevich cover whose Čech nerve consists entirely of stacks with the resolution property [Hoy17, Lemma 3.11]. In particular, the requirement that stacks in tqStk_B have the resolution property is immaterial as far as Nisnevich sheaves are concerned. Allowing $\mathbf{B}G$-stacks that are only *Nisnevich-locally* quasi-projective is necessary to dispense with isotriviality conditions on G when the base B is not geometrically unibranch (see [Hoy17, Remark 2.9]).

is cartesian. For quotients by finite discrete groups, we can improve this result as follows:

Theorem 1.5. *Let G be a finite discrete group and let*

$$\begin{tikzcd} W \arrow[r, hook] \arrow[d] & Y \arrow[d, "p"] \\ Z \arrow[r, hook, "i"'] & X \end{tikzcd}$$

be a cartesian square of locally affine qcs G-schemes over $\mathbb{Z}[1/|G|]$, where i is a closed immersion, p is proper, and p induces an isomorphism $Y \smallsetminus W \simeq X \smallsetminus Z$. Then the induced square of spectra

$$\begin{tikzcd} KH([X/G]) \arrow[r] \arrow[d] & KH([Z/G]) \arrow[d] \\ KH([Y/G]) \arrow[r] & KH([W/G]) \end{tikzcd}$$

is cartesian, where $KH(\mathfrak{X})$ denotes the geometric realization of the simplicial spectrum $K^B(\Delta^\bullet \times \mathfrak{X})$.

Remark 1.6. If G is a finite discrete group acting on a qcs scheme X, then X is a locally affine G-scheme if and only if the coarse moduli space of the Deligne–Mumford stack $[X/G]$ is a scheme [Ryd13, Remark 4.5].

We make a few comments on homotopy invariance. As we observed in [Hoy17], most of the interesting properties of homotopy invariant Nisnevich sheaves on schemes only extend to stacks if homotopy invariance is understood in the strong sense of property (2) of Theorem 1.3. A typical example of a homotopy equivalence in that sense is the quotient map $\mathfrak{X} \to \mathfrak{X}/U$ where U is a split unipotent group acting on $\mathfrak{X}$; this map is usually not an $\mathbb{A}^1$-homotopy equivalence, not even Nisnevich-locally on the target. This explains why our definition of KH for general stacks is more complicated than it is for schemes. Property (4) of Theorem 1.3 is explained by the fact that vector bundle torsors over such stacks are Nisnevich-locally split.

Properties (1)–(4) of Theorem 1.3 will essentially be enforced by the definition of the homotopy K-theory presheaf KH and foundational results on equivariant K-theory due to Thomason [Tho87] and Krishna–Ravi [KR18]. The content of Theorem 1.3 is thus the statement that KH is a cdh sheaf. Its proof uses the machinery of stable equivariant motivic homotopy theory developed in [Hoy17]. Namely, the fact that KH is a Nisnevich sheaf satisfying properties (2) and (3) of Theorem 1.3 implies that its restriction to smooth N-quasi-projective $\mathfrak{X}$-stacks is representable by a motivic spectrum $\mathrm{KGL}_\mathfrak{X} \in \mathrm{SH}(\mathfrak{X})$. By [Hoy17, Corollary 6.25], we can then deduce that KH satisfies cdh descent, provided that the family of motivic spectra $\{\mathrm{KGL}_\mathfrak{X}\}_{\mathfrak{X} \in \mathrm{tqStk}_B}$ is stable under N-quasi-projective base change. This base change property is thus the heart of the proof. We will verify it by adapting Morel and Voevodsky's geometric construction of classifying spaces [MV99, §4.2] to the equivariant setting.

Theorem 1.7. *For every $\mathfrak{X} \in \mathrm{tqStk}_B$, there exists an E_∞-algebra $\mathrm{KGL}_\mathfrak{X} \in \mathrm{SH}(\mathfrak{X})$ representing the E_∞-ring-valued presheaf KH on smooth N-quasi-projective $\mathfrak{X}$-stacks. Moreover, the assignment $\mathfrak{X} \mapsto \mathrm{KGL}_\mathfrak{X}$ is a section of $\mathrm{CAlg}(\mathrm{SH}(-))$ over $\mathrm{tqStk}_B^{\mathrm{op}}$ that is cocartesian over N-quasi-projective morphisms. In particular, for $f\colon \mathfrak{Y} \to \mathfrak{X}$ N-quasi-projective, $f^*(\mathrm{KGL}_\mathfrak{X}) \simeq \mathrm{KGL}_\mathfrak{Y}$.*

Finally, we will observe that the Borel–Moore homology theory represented by $\mathrm{KGL}_\mathfrak{X}$ on N-quasi-projective $\mathfrak{X}$-stacks, for $\mathfrak{X}$ regular, is the K-theory of coherent sheaves, also known as G-theory.

Remark 1.8. In the paper [KR18], the authors work over a base field. This assumption is used via [HR15] to ensure that the ∞-category $\mathrm{QCoh}(\mathfrak{X})$ of quasi-coherent sheaves is compactly generated and that the structure sheaf $\mathcal{O}_\mathfrak{X}$ is compact. We claim that this holds for any $\mathfrak{X} \in \mathrm{tqStk}_B$. If G is a linearly reductive affine group scheme, then $\mathcal{O}_{\mathbf{B}G}$ is compact in $\mathrm{QCoh}(\mathbf{B}G)$, by [HR15, Theorem C (3)⇒(1)]. If $\mathfrak{X}$ is N-quasi-projective over $\mathbf{B}G$, then $p\colon \mathfrak{X} \to \mathbf{B}G$ is representable, so the functor $p^*\colon \mathrm{QCoh}(\mathbf{B}G) \to \mathrm{QCoh}(\mathfrak{X})$ preserves compact objects. Hence, locally free modules of finite rank over $\mathfrak{X}$ are compact, being dualizable. Finally, as $\mathfrak{X}$ has the resolution property, $\mathrm{QCoh}(\mathfrak{X})$ is generated under colimits by shifts of locally free modules of finite rank, by [Lur18, Proposition 9.3.3.7, Corollary C.2.1.7, and Corollary 9.1.3.2 (4)]. Thus, we shall freely use the results of [KR18] over a general qcs base scheme B.

Outline. In §2, we construct geometric models for classifying spaces of linear algebraic group in equivariant motivic homotopy theory. The main example is a model for the classifying space of GL_n in terms of equivariant Grassmannians.

In §3, we develop some categorical machinery that will be used to equip the motivic spectrum $\mathrm{KGL}_{\mathfrak{X}}$ with an E_∞-ring structure. The results of this section are not otherwise essential for the proof of Theorem 1.3, but they may be of independent interest.

In §4, we define homotopy K-theory of tame quotient stacks and prove that it satisfies properties (1)–(4) of Theorem 1.3.

In §5, we construct the motivic E_∞-ring spectra $\mathrm{KGL}_{\mathfrak{X}}$ representing homotopy K-theory and prove that they are stable under N-quasi-projective base change, which implies that KH is a cdh sheaf.

Notation and terminology. This paper is a sequel to [Hoy17] and uses many of the definitions and constructions introduced there, such as: the notions of homotopy invariance and Nisnevich excision [Hoy17, Definitions 3.3 and 3.7], the corresponding localization functors L_{htp} and L_{Nis}, and the combined motivic localization L_{mot} [Hoy17, §3.4]; the auxiliary notion of small G-scheme [Hoy17, Definition 3.1]; and the definitions of the stable equivariant motivic homotopy category as a symmetric monoidal ∞-category and as an ∞-category of spectrum objects [Hoy17, §6.1]. A notational difference with *op. cit.* is that we prefer to work with stacks rather than G-schemes, so that we write, e.g., $\mathrm{SH}([X/G])$ instead of $\mathrm{SH}^G(X)$.

Given $\mathfrak{X} \in \mathrm{tqStk}_B$, recall that $\mathrm{Sch}_{\mathfrak{X}} \subset (\mathrm{tqStk}_B)_{/\mathfrak{X}}$ is the full subcategory of N-quasi-projective $\mathfrak{X}$-stacks. Whenever we write $\mathfrak{X}$ as $[X/G]$, it is understood that G is a tame affine group scheme and that $\mathfrak{X} \in \mathrm{Sch}_{\mathbf{B}G}$. If $\mathfrak{X} = [X/G]$, $\mathrm{Sch}_{\mathfrak{X}}$ differs slightly from the category Sch_X^G from [Hoy17, §3.1], but every object in either category has a Nisnevich cover whose Čech nerve belongs to their intersection, so the difference does not matter for our purposes. We let $\mathrm{Sm}_{\mathfrak{X}} \subset \mathrm{Sch}_{\mathfrak{X}}$ be the full subcategory spanned by the smooth $\mathfrak{X}$-stacks. We denote by $\mathrm{QCoh}(\mathfrak{X})^\heartsuit$ the abelian category of quasi-coherent sheaves on $\mathfrak{X}$ (it is the heart of a t-structure on the stable ∞-category $\mathrm{QCoh}(\mathfrak{X})$ from Remark 1.8). Given $\mathcal{E} \in \mathrm{QCoh}(\mathfrak{X})^\heartsuit$, we denote by $\mathbb{V}(\mathcal{E}) = \mathrm{Spec}(\mathrm{Sym}(\mathcal{E}))$ the associated vector bundle and by $\mathbb{P}(\mathcal{E}) = \mathrm{Proj}(\mathrm{Sym}(\mathcal{E}))$ the associated projective bundle. Unless otherwise specified, presheaves and sheaves are valued in ∞-groupoids.

2. Geometric models for equivariant classifying spaces

In this section, we fix a base stack $\mathfrak{S} = [S/G] \in \mathrm{tqStk}_B$. If Γ is an fppf sheaf of groups on $\mathrm{Sch}_{\mathfrak{S}}$, we denote by $B_{\mathrm{fppf}}\Gamma = L_{\mathrm{fppf}}(*/\Gamma)$ the presheaf of groupoids classifying Γ-torsors in the fppf topology, which we will often implicitly regard as a presheaf on $\mathrm{Sm}_{\mathfrak{S}}$ (note however that the fppf sheafification must be performed on the larger category $\mathrm{Sch}_{\mathfrak{S}}$). For example, for $\mathfrak{X} \in \mathrm{Sm}_{\mathfrak{S}}$ and $n \geq 0$, $(B_{\mathrm{fppf}}\mathrm{GL}_n)(\mathfrak{X})$ is the groupoid of vector bundles of rank n on $\mathfrak{X}$. When $\mathfrak{S}$ is a scheme and Γ is a smooth linear group scheme over $\mathfrak{S}$, Morel and Voevodsky constructed in [MV99, §4.2] a geometric model for $L_{\mathrm{mot}}(B_{\mathrm{fppf}}\Gamma)$, i.e., they expressed $L_{\mathrm{mot}}(B_{\mathrm{fppf}}\Gamma)$ as a simple colimit of representables in $\mathrm{H}(\mathfrak{S})$. In this section, we generalize their result to arbitrary $\mathfrak{S} \in \mathrm{tqStk}_B$.

Let U be an fppf sheaf on $\mathrm{Sch}_{\mathfrak{S}}$ with an action of Γ. If X is an fppf sheaf and $\pi\colon T \to X$ is a torsor under Γ, we denote by U_π the π-twisted form of U, i.e., the sheaf $L_{\mathrm{fppf}}((U \times T)/\Gamma)$. The Morel–Voevodsky construction is based on the following tautological lemma:

Lemma 2.1. *Let Γ be an fppf sheaf of groups on $\mathrm{Sch}_{\mathfrak{S}}$ acting on an fppf sheaf U. Suppose that, for every $\mathfrak{X} \in \mathrm{Sm}_{\mathfrak{S}}$ and every fppf torsor $\pi\colon T \to \mathfrak{X}$ under Γ, $U_\pi \to \mathfrak{X}$ is a motivic equivalence on $\mathrm{Sm}_{\mathfrak{S}}$. Then the map*

$$L_{\mathrm{fppf}}(U/\Gamma) \to B_{\mathrm{fppf}}\Gamma$$

induced by $U \to *$ is a motivic equivalence on $\mathrm{Sm}_{\mathfrak{S}}$.

Proof. By universality of colimits, it suffices to show that, for every $\mathfrak{X} \in \mathrm{Sm}_{\mathfrak{S}}$ and every map $\mathfrak{X} \to L_{\mathrm{fppf}}(*/\Gamma)$, the projection $L_{\mathrm{fppf}}(U/\Gamma \times_{*/\Gamma} \mathfrak{X}) \to \mathfrak{X}$ is a motivic equivalence on $\mathrm{Sm}_{\mathfrak{S}}$. This is exactly the assumption. □

Recall from [Hoy17, Definition 3.1] that a G-scheme X over B is *small* if there exists a G-quasi-projective morphism $X \to U$ where U is affine, has trivial G-action, and has the G-resolution property. Every $\mathfrak{X} \in \mathrm{tqStk}_B$ admits a schematic Nisnevich cover whose Čech nerve consists entirely of stacks of the form $[X/G]$ where X is small [Hoy17, Lemma 3.11].

Definition 2.2. A *system of vector bundles* over $\mathfrak{S}$ is a diagram of vector bundles $(V_i)_{i\in I}$ over $\mathfrak{S}$, where I is a filtered poset, whose transition maps are vector bundle inclusions. Such a system is called:

- *saturated* if, for every $i \in I$, there exists $2i \geq i$ such that $V_i \hookrightarrow V_{2i}$ is isomorphic under V_i to $(\mathrm{id}, 0)\colon V_i \hookrightarrow V_i \times_{\mathfrak{S}} V_i$.
- *complete* if, for every $\mathfrak{X} = [X/G] \in \mathrm{Sch}_{\mathfrak{S}}$ with X small and affine, and for every vector bundle E on $\mathfrak{X}$, there exists $i \in I$ and a vector bundle inclusion $E \hookrightarrow V_i \times_{\mathfrak{S}} \mathfrak{X}$.[2]

Note that both properties are preserved by any base change $\mathfrak{T} \to \mathfrak{S}$ in $\mathrm{Sch}_{\mathfrak{S}}$. The following example shows that complete saturated systems of vector bundles always exist.

Example 2.3.

(1) If G is finite locally free and $p\colon S \to \mathfrak{S} = [S/G]$ is the quotient map, then $(\mathbb{V}(p_*\mathcal{O}_S^n))_{n\geq 0}$ is a complete saturated system of vector bundles over $\mathfrak{S}$.

(2) Let $\{V_\alpha\}_{\alpha\in A}$ be a set of representatives of isomorphism classes of vector bundles over $\mathfrak{S}$, let I be the filtered poset of maps $A \to \mathbb{N}$ with finitely many nonzero values, and for $i \in I$ let $V_i = \bigoplus_{\alpha\in A} V_\alpha^{i_\alpha}$. Then $(V_i)_{i\in I}$, with the obvious transition maps, is clearly a saturated system of vector bundles over $\mathfrak{S}$. It is also complete, by Lemma 2.4 below.

Lemma 2.4. *Let $f\colon \mathfrak{T} \to \mathfrak{S}$ be a quasi-affine morphism. For every vector bundle V on $\mathfrak{T}$, there exists a vector bundle W on $\mathfrak{S}$ and a vector bundle inclusion $V \hookrightarrow W \times_{\mathfrak{S}} \mathfrak{T}$.*

Proof. Let $V = \mathbb{V}(\mathcal{E})$. Since f is quasi-affine, $f^*f_*(\mathcal{E}) \to \mathcal{E}$ is an epimorphism. Since $f_*(\mathcal{E})$ is the union of its finitely generated quasi-coherent submodules [Hoy17, Lemma 2.10], there exists $\mathcal{M} \subset f_*(\mathcal{E})$ finitely generated such that $f^*(\mathcal{M}) \to \mathcal{E}$ is an epimorphism. By the resolution property, we may assume that $\mathcal{M}$ is locally free. Setting $W = \mathbb{V}(\mathcal{M})$, we then have a vector bundle inclusion $V \hookrightarrow W \times_{\mathfrak{S}} \mathfrak{T}$, as desired. □

Lemma 2.5. *Let $\mathfrak{X} = [X/G] \in \mathrm{tqStk}_B$ with X small and affine, let $s\colon \mathfrak{Z} \hookrightarrow \mathfrak{X}$ be a closed immersion, and let V be a vector bundle on $\mathfrak{X}$. Then any section of V over $\mathfrak{Z}$ lifts to a section of V over $\mathfrak{X}$.*

Proof. Let $V = \mathbb{V}(\mathcal{E})$. We must show that any map $\mathcal{O}_{\mathfrak{X}} \to s_*s^*(\mathcal{E}^\vee)$ in $\mathrm{QCoh}(\mathfrak{X})^\heartsuit$ lifts to a map $\mathcal{O}_{\mathfrak{X}} \to \mathcal{E}^\vee$. Since s is a closed immersion, the restriction map $\mathcal{E}^\vee \to s_*s^*(\mathcal{E}^\vee)$ is an epimorphism in $\mathrm{QCoh}(\mathfrak{X})^\heartsuit$. Moreover, since X is small and affine and G is linearly reductive, $\mathcal{O}_{\mathfrak{X}}$ is projective in $\mathrm{QCoh}(\mathfrak{X})^\heartsuit$ [Hoy17, Lemma 2.17]. The result follows. □

Lemma 2.6. *Let $(V_i)_{i\in I}$ be a saturated system of vector bundles over $\mathfrak{S}$. For every $i \in I$, let $U_i \subset V_i$ be an open substack such that $V_i \hookrightarrow V_j$ maps U_i to U_j whenever $i \leq j$. Suppose that:*

(1) there exists $i \in I$ such that $U_i \to \mathfrak{S}$ has a section;

(2) for all $i \in I$, under the isomorphism $V_{2i} \simeq V_i^2$, $(U_i \times V_i) \cup (V_i \times U_i) \subset U_{2i}$.

Then $U_\infty = \mathrm{colim}_{i\in I} U_i \in \mathcal{P}(\mathrm{Sm}_{\mathfrak{S}})$ is motivically contractible.

Proof. By [Hoy17, Proposition 3.16 (2)], it will suffice to show that, for every $\mathfrak{X} = [X/G] \in \mathrm{Sm}_{\mathfrak{S}}$ with X small and affine, the simplicial set $\mathrm{Map}(\mathbb{A}^\bullet \times \mathfrak{X}, U_\infty)$ is a contractible Kan complex. Consider a lifting problem

$$\begin{tikzcd} \partial\Delta^n \arrow[r, "f"] \arrow[d, hook] & \mathrm{Map}(\mathbb{A}^\bullet \times \mathfrak{X}, U_\infty). \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd}$$

Then $f\colon \partial\mathbb{A}^n_{\mathfrak{X}} \to U_\infty$ is a morphism from the boundary of the algebraic n-simplex over $\mathfrak{X}$ to U_∞, and it factors through U_i for some i since $\partial\mathbb{A}^n_{\mathfrak{X}}$ is compact as an object of $\mathcal{P}(\mathrm{Sm}_{\mathfrak{S}})$. Increasing i if necessary, we may assume, by (1), that there exists a section $x\colon \mathfrak{S} \to U_i$. By Lemma 2.5, there exists a morphism $g\colon \mathbb{A}^n_{\mathfrak{X}} \to V_i$ lifting f. Choose a closed substack $Z_i \subset V_i$ complementary to U_i, so that $g^{-1}(Z_i) \cap \partial\mathbb{A}^n_{\mathfrak{X}} = \emptyset$. Again by Lemma 2.5, the map

$$g^{-1}(Z_i) \sqcup \partial\mathbb{A}^n_{\mathfrak{X}} \to \mathfrak{S} \sqcup \mathfrak{S} \xrightarrow{x \sqcup 0} V_i$$

admits an extension $h\colon \mathbb{A}^n_{\mathfrak{X}} \to V_i$. Then the morphism $(g, h)\colon \mathbb{A}^n_{\mathfrak{X}} \to V_i^2$ misses Z_i^2 and hence solves the lifting problem, by (2). □

[2] This is similar to the notion of *complete G-universe* in equivariant homotopy theory.

If E is a vector bundle over $\mathfrak{S}$, we denote by $\mathrm{GL}(E)$ the group $\mathfrak{S}$-stack of linear automorphisms of E. By a *subgroup* of $\mathrm{GL}(E)$ we mean a subfunctor of its functor of points (valued in group objects in groupoids).

Theorem 2.7. *Let E be a vector bundle over $\mathfrak{S}$, $\Delta \subset \mathrm{GL}(E)$ a closed subgroup, and $\Gamma \subset \Delta$ a subgroup that is flat and finitely presented over $\mathfrak{S}$. Let $(V_i)_{i\in I}$ be a complete saturated system of vector bundles over $\mathfrak{S}$. For each $i \in I$, let $U_i \subset \mathrm{Hom}(E, V_i)$ be the open substack where the action of Δ is strictly free, and let $U_\infty = \mathrm{colim}_{i\in I} U_i$. Then the map*

$$L_{\mathrm{fppf}}(U_\infty/\Gamma) \to B_{\mathrm{fppf}}\Gamma$$

induced by $U_\infty \to *$ is a motivic equivalence on $\mathrm{Sm}_{\mathfrak{S}}$.

Proof. We check that U_∞ satisfies the assumption of Lemma 2.1, i.e., that for any $\mathfrak{X} \in \mathrm{Sm}_{\mathfrak{S}}$ and any Γ-torsor $\pi\colon T \to \mathfrak{X}$, the map $(U_\infty)_\pi \to \mathfrak{X}$ is a motivic equivalence on $\mathrm{Sm}_{\mathfrak{S}}$. By [Hoy17, Proposition 4.6], we can assume that $\mathfrak{X} = [X/G]$ with X small and affine. It then suffices to show that the saturated system of vector bundles $\mathrm{Hom}(E_\pi, V_i \times_{\mathfrak{S}} \mathfrak{X})$ over $\mathfrak{X}$ and the open substacks $(U_i)_\pi$ satisfy the conditions of Lemma 2.6 with $\mathfrak{S} = \mathfrak{X}$. The second condition is clear, by definition of U_i. To verify the first condition, we can assume that $\Delta = \mathrm{GL}(E)$. Sections of $(U_i)_\pi$ over $\mathfrak{X}$ are then vector bundle inclusions $E_\pi \hookrightarrow V_i \times_{\mathfrak{S}} \mathfrak{X}$. Since $(V_i)_{i\in I}$ is complete, there exist such inclusions for large enough i. □

Remark 2.8. Although this is not always true in the generality of Theorem 2.7, the fppf quotients $L_{\mathrm{fppf}}(U_i/\Gamma)$ are often representable by (necessarily smooth) quasi-projective $\mathfrak{S}$-stacks, so that the presheaf $L_{\mathrm{fppf}}(U_\infty/\Gamma)$ is a filtered colimit of representables. It is in that sense that $L_{\mathrm{fppf}}(U_\infty/\Gamma)$ is a geometric model for $L_{\mathrm{mot}}(B_{\mathrm{fppf}}\Gamma)$.

Corollary 2.9. *Under the assumptions of Theorem 2.7, suppose that the fppf quotients $L_{\mathrm{fppf}}(U_i/\Gamma)$ are universally representable by N-quasi-projective $\mathfrak{S}$-stacks. Then, for every N-quasi-projective morphism $f\colon \mathfrak{T} \to \mathfrak{S}$, the map*

$$f^*(B_{\mathrm{fppf}}\Gamma) \to B_{\mathrm{fppf}}(f^*\Gamma)$$

in $\mathcal{P}(\mathrm{Sm}_{\mathfrak{T}})$ is a motivic equivalence.

Proof. Consider the following commutative square in $\mathcal{P}(\mathrm{Sm}_{\mathfrak{T}})$:

$$\begin{tikzcd}
f^*L_{\mathrm{fppf}}(U_\infty/\Gamma) \arrow[r] \arrow[d] & f^*(B_{\mathrm{fppf}}\Gamma) \arrow[d] \\
L_{\mathrm{fppf}}(f^*(U_\infty/\Gamma)) \arrow[r] & B_{\mathrm{fppf}}(f^*\Gamma).
\end{tikzcd}$$

By Theorem 2.7, the horizontal maps are motivic equivalences. On the other hand, by assumption, the left vertical arrow is an isomorphism between ind-representable presheaves on $\mathrm{Sm}_{\mathfrak{T}}$. □

Corollary 2.9, applied to $\Gamma = \mathrm{GL}_n$, is all that we will need from this section in the sequel. In that case, $U_i \subset \mathrm{Hom}(\mathbb{A}^n_{\mathfrak{S}}, V_i)$ is the open substack of vector bundle inclusions, and $L_{\mathrm{fppf}}(U_i/\mathrm{GL}_n)$ is universally represented by the Grassmannian $\mathrm{Gr}_n(V_i)$. Let us make Theorem 2.7 more explicit in this special case:

Corollary 2.10. *Let $(V_i)_{i\in I}$ be a complete saturated system of vector bundles over $\mathfrak{S}$. For any $n \geq 0$, the map*

$$\operatorname*{colim}_{i\in I} \mathrm{Gr}_n(V_i) \to B_{\mathrm{fppf}}\mathrm{GL}_n$$

in $\mathcal{P}(\mathrm{Sm}_{\mathfrak{S}})$ classifying the tautological bundles is a motivic equivalence.

3. Periodic E_∞-algebras

Let $\mathcal{C} \in \mathrm{CAlg}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$ be a presentably symmetric monoidal ∞-category, S a set of objects of $\mathcal{C}_{/\mathbf{1}}$, and $\mathcal{M}$ a $\mathcal{C}$-module in $\mathcal{P}\mathrm{r}^{\mathrm{L}}$. For every $x \in \mathcal{C}$, we have the adjunction

$$x \otimes - : \mathcal{M} \rightleftarrows \mathcal{M} : \mathrm{Hom}(x, -).$$

We say that an object $E \in \mathcal{M}$ is *S-periodic* if $\alpha^*\colon E \to \mathrm{Hom}(x, E)$ is an equivalence for every $\alpha\colon x \to \mathbf{1}$ in S. We denote by $P_S\mathcal{M} \subset \mathcal{M}$ the full subcategory spanned by the S-periodic objects. It is clear that this inclusion is an accessible localization and hence admits a left adjoint P_S, called *periodization*. Note that E is S-periodic if and only if it is local with respect to $\mathrm{id}_M \otimes \alpha$ for every $M \in \mathcal{M}$ and $\alpha \in S$. If $\mathcal{M} = \mathcal{C}$, it follows immediately that the localization functor P_S is compatible with the monoidal structure in the sense of [Lur17, Definition 2.2.1.6], and hence that it can be promoted to a symmetric monoidal functor [Lur17, Proposition 2.2.1.9]. In particular, for every E_∞-algebra A in $\mathcal{C}$, P_SA is also an E_∞-algebra in $\mathcal{C}$ and $A \to P_SA$ is an E_∞-map.

Let S_0 be the set of domains of morphisms in S. Consider the presentably symmetric monoidal ∞-category $\mathcal{C}[S_0^{-1}]$ obtained from $\mathcal{C}$ by adjoining formal inverses to elements of S_0 (see [Hoy17, §6.1]), which is in particular a $\mathcal{C}$-module. We have an adjunction

$$\mathcal{C} \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} \mathcal{C}[S_0^{-1}],$$

where Φ is symmetric monoidal. It follows that Ψ preserves S-periodic objects. Hence, the above adjunction induces an adjunction

$$P_S\mathcal{C} \underset{\Psi}{\overset{P_S\Phi}{\rightleftarrows}} P_S(\mathcal{C}[S_0^{-1}]). \tag{3.1}$$

Proposition 3.2. *Let $\mathcal{C}$ be a presentably symmetric monoidal ∞-category, S a set of objects of $\mathcal{C}_{/\mathbf{1}}$, and S_0 the set of domains of morphisms in S. Then the adjunction* (3.1) *is an equivalence of symmetric monoidal ∞-categories. In particular, every S-periodic E_∞-algebra in $\mathcal{C}$ lifts uniquely to an S-periodic E_∞-algebra in $\mathcal{C}[S_0^{-1}]$.*

Proof. Indeed, the symmetric monoidal functors $P_S\colon \mathcal{C} \to P_S\mathcal{C}$ and $P_S\Phi\colon \mathcal{C} \to P_S(\mathcal{C}[S_0^{-1}])$ satisfy the same universal property, since the former sends every $x \in S_0$ to an invertible object, namely, the unit of $P_S\mathcal{C}$. □

We would like to understand the periodization functor P_S more explicitly. Consider the case where S consists of a single map $\alpha\colon x \to \mathbf{1}$. Given $E \in \mathcal{C}$, it is tempting to think that $P_\alpha E$ is given by the formula

$$\operatorname{colim}\bigl(E \xrightarrow{\alpha} \operatorname{Hom}(x, E) \xrightarrow{\alpha} \operatorname{Hom}(x^{\otimes 2}, E) \xrightarrow{\alpha} \cdots\bigr),$$

at least if we assume that $\operatorname{Hom}(x, -)$ preserves filtered colimits (otherwise, we would naturally consider a transfinite construction). This formula is indeed correct if $\mathcal{C}$ is a stable ∞-category and $\alpha\colon \mathbf{1} \to \mathbf{1}$ is multiplication by an integer, but not in general. For example, suppose that $\mathcal{C}$ is the symmetric monoidal ∞-category of small stable ∞-categories, and let α be multiplication by a positive integer on the unit $\mathrm{Sp}^{\mathrm{fin}}$. Then $P_\alpha\mathcal{C} \subset \mathcal{C}$ is the subcategory of zero objects, but the above colimit with $E = \mathrm{Sp}^{\mathrm{fin}}$ is not zero. The essential difference between these two cases is the following: in the first case, the cylic permutation of α^3 is homotopic to the identity (because it is the image of an even element in π_1 of the sphere spectrum), but in the second case, no nontrivial permutation of α^n is homotopic to the identity. We will show that there exists an analogous formula for P_S in general, provided that the elements of S are cyclically symmetric in a suitable sense.

We recall some constructions from [Hoy17, §6.1]. Let X be any set of objects of $\mathcal{C}$. The filtered simplicial set $L(X)$ is the union over finite subsets $F \subset X$ of the simplicial sets L^F, where L is the 1-skeleton of the nerve of the poset $\mathbb{N}$. We view a vertex of $L(X)$ as a formal tensor product of elements of X. The $\mathcal{C}$-module $\mathrm{Stab}_X(\mathcal{C})$ of *X-spectra* is then defined as the limit of a diagram $L(X)^{\mathrm{op}} \to \mathrm{Mod}_{\mathcal{C}}$ taking each vertex of $L(X)$ to $\mathcal{C}$ and each arrow $w \to w \otimes x$ to the functor $\operatorname{Hom}(x, -)$. Equivalently, $\mathrm{Stab}_X(\mathcal{C})$ is the ∞-category of cartesian sections of the cartesian fibration over $L(X)$ classified by $L(X)^{\mathrm{op}} \to \mathcal{C}\mathrm{at}_\infty$. A general section of this cartesian fibration will be called an *X-prespectrum* in $\mathcal{C}$; we denote by $\mathrm{Stab}_X^{\mathrm{lax}}(\mathcal{C})$ the $\mathcal{C}$-module of X-prespectra. Thus, $\mathrm{Stab}_X(\mathcal{C})$ is a (left exact) localization of $\mathrm{Stab}_X^{\mathrm{lax}}(\mathcal{C})$. The localization functor is called *spectrification* and is denoted by $Q\colon \mathrm{Stab}_X^{\mathrm{lax}}(\mathcal{C}) \to \mathrm{Stab}_X(\mathcal{C})$. If $\operatorname{Hom}(x, -)$ preserves filtered colimits for all $x \in X$, which will be the case in all our applications, spectrification is given by the familiar formula

$$Q(E)_w = \operatorname*{colim}_{v \in L(X)} \operatorname{Hom}(v, E_{w \otimes v}).$$

In general, one can describe spectrification as follows. For every $x \in X$, consider the full subcategory $\mathcal{E}_x \subset \mathrm{Stab}_X^{\mathrm{lax}}(\mathcal{C})$ consisting of X-prespectra that are spectra in the x-direction, so that $\mathrm{Stab}_X(\mathcal{C}) = \bigcap_{x \in X} \mathcal{E}_x$. Choose a regular cardinal κ such that $\operatorname{Hom}(x, -)$ preserves κ-filtered colimits for all $x \in X$, and let sh_x be the pointed endofunctor of $\mathrm{Stab}_X^{\mathrm{lax}}(\mathcal{C})$ given by $\mathrm{sh}_x(E)_w = \operatorname{Hom}(x, E_{w \otimes x})$. Then the κth iteration sh_x^κ of sh_x lands in $\mathcal{E}_x$. Moreover, any map $E \to F$ with $F \in \mathcal{E}_x$ factors uniquely through $\mathrm{sh}_x(E)$. It follows that sh_x^κ is left adjoint to the inclusion $\mathcal{E}_x \subset \mathrm{Stab}_X^{\mathrm{lax}}(\mathcal{C})$. The total localization functor Q can now be written as an appropriate κ-filtered transfinite composition in which each indecomposable map is an instance of $\mathrm{id} \to \mathrm{sh}_x^\kappa$ for some $x \in X$ (see the proof of [Lur09, Lemma 7.3.2.3]).

To every $E \in \mathcal{C}$ we can associate a "constant" S_0-prespectrum $c_S E = (E)_{w \in L(S_0)}$ with structure maps $E \to \mathrm{Hom}(x, E)$ induced by the maps in S. Let $Q_S \colon \mathcal{C} \to \mathcal{C}$ be the functor defined by

$$Q_S E = \Omega^\infty Q(c_S E),$$

where $\Omega^\infty \colon \mathrm{Stab}_{S_0}(\mathcal{C}) \to \mathcal{C}$ is evaluation at the initial vertex of $L(S_0)$. There is an obvious natural transformation $\mathrm{id} \to Q_S$. For example, if S consists of a single map $\alpha \colon x \to \mathbf{1}$ and $\mathrm{Hom}(x, -)$ preserves filtered colimits, we have

$$Q_\alpha E = \mathrm{colim}\big(E \xrightarrow{\alpha} \mathrm{Hom}(x, E) \xrightarrow{\alpha} \mathrm{Hom}(x^{\otimes 2}, E) \to \cdots\big).$$

Lemma 3.3. *Let $\mathcal{C}$ be a presentably symmetric monoidal ∞-category, S a set of objects of $\mathcal{C}_{/\mathbf{1}}$, and $E \in \mathcal{C}$. If $Q_S E$ is S-periodic, then the map $E \to Q_S E$ exhibits $Q_S E$ as the S-periodization of E.*

Proof. For $x \in S_0$, the functor $\mathrm{Hom}(x, -) \colon P_S\mathcal{C} \to P_S\mathcal{C}$ is an equivalence of ∞-categories, since $P_S(x)$ is invertible in $P_S\mathcal{C}$. Hence, $\Omega^\infty \colon P_S \mathrm{Stab}_{S_0}(\mathcal{C}) \to P_S\mathcal{C}$ is an equivalence. Consider the following commutative diagram of $\mathcal{C}$-modules:

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "\Sigma^\infty_{\mathrm{lax}}"'] \arrow[rr, bend left, "\Sigma^\infty"] \arrow[d] & \mathrm{Stab}^{\mathrm{lax}}_{S_0}(\mathcal{C}) \arrow[r, "Q"'] \arrow[d] & \mathrm{Stab}_{S_0}(\mathcal{C}) \arrow[d] & \\
P_S\mathcal{C} \arrow[r] \arrow[rr, bend right, "\simeq"'] & P_S \mathrm{Stab}^{\mathrm{lax}}_{S_0}(\mathcal{C}) \arrow[r] & P_S \mathrm{Stab}_{S_0}(\mathcal{C}) \arrow[r, "\Omega^\infty", "\simeq"'] & P_S\mathcal{C}.
\end{tikzcd}$$

All the vertical arrows are periodization functors, and the lower composition is the identity. This diagram shows that

$$P_S(E) = \Omega^\infty P_S Q(\Sigma^\infty_{\mathrm{lax}} E).$$

Here, $\Sigma^\infty_{\mathrm{lax}} E$ is the free S_0-prespectrum $(E \otimes w)_{w \in L(S_0)}$. The obvious map $\Sigma^\infty_{\mathrm{lax}} E \to c_S E$ is manifestly a termwise P_S-equivalence. Since the right adjoints to the various evaluation functors $\mathrm{Stab}^{\mathrm{lax}}_{S_0}(\mathcal{C}) \to \mathcal{C}$ preserve S-periodic objects, termwise P_S-equivalences of S_0-prespectra are in fact P_S-equivalences. It follows that

$$P_S(E) = \Omega^\infty P_S Q(c_S E).$$

All the terms of the S_0-spectrum $Q(c_S E)$ are equivalent to $Q_S E$. Hence, by the assumption, $Q(c_S E)$ is already S-periodic, and we get $P_S E = Q_S E$, as desired. $\square$

Example 3.4. Let K denote the presheaf of E_∞-ring spectra $X \mapsto K(X)$ on qcqs schemes, and let $\beta \in \tilde{K}_1(\mathbb{G}_m, 1)$ be the *Bott element*, that is, the element induced by the automorphism t of $\mathcal{O}_{\mathbb{G}_m}$, where $\mathbb{G}_m = \mathrm{Spec}\,\mathbb{Z}[t^{\pm 1}]$. Let γ be the composite

$$(\mathbb{P}^1 \smallsetminus 0) \coprod_{\mathbb{G}_m} \mathbb{A}^1 \to \Sigma(\mathbb{G}_m/1) \xrightarrow{\beta} K,$$

where the pushout is taken in presheaves and pointed at 1. By inspecting the definition [TT90, Definition 6.4], we see that the Bass–Thomason–Trobaugh K-theory spectrum K^B is the K-module $Q_\gamma K$. Since K^B is γ-periodic, Lemma 3.3 implies that $K^B = P_\gamma K$. In particular, K^B is an E_∞-algebra under K. The same argument applies to K and K^B as presheaves on tqStk_B (see [KR18, §3.5] for the definition of K^B in this context).

Let $\mathcal{C}$ be a symmetric monoidal ∞-category. An object $x \in \mathcal{C}$ is called *n-symmetric* if the cyclic permutation σ_n of $x^{\otimes n}$ is homotopic to the identity. We will say that x is *symmetric* if it is n-symmetric for some $n \geq 2$. If $\mathcal{C}$ is presentably symmetric monoidal and X is a set of symmetric objects of $\mathcal{C}$, there is an equivalence of $\mathcal{C}$-modules $\mathcal{C}[X^{-1}] \simeq \mathrm{Stab}_X(\mathcal{C})$ (see [Rob15, Corollary 2.22] and [Hoy17, §6.1]).

The ∞-category $\mathcal{C}_{/\mathbf{1}}$ inherits a symmetric monoidal structure from $\mathcal{C}$ such that the forgetful functor $\mathcal{C}_{/\mathbf{1}} \to \mathcal{C}$ is symmetric monoidal. An n-symmetric object in $\mathcal{C}_{/\mathbf{1}}$ is then a morphism $\alpha \colon x \to \mathbf{1}$ such that the cyclic permutation σ_n of $x^{\otimes n}$ is homotopic *over* $\mathbf{1}$ to the identity.

Example 3.5. If $\mathcal{C}$ is symmetric monoidal, $\mathrm{End}(\mathbf{1})$ is an E_∞-space under composition. In particular, for every $\alpha\colon \mathbf{1}\to\mathbf{1}$, the cyclic permutation of n letters induces a self-homotopy σ_n of α^n. Then α is n-symmetric in $\mathcal{C}_{/\mathbf{1}}$ if and only if the homotopy class of σ_n is in the image of the group homomorphism $\pi_1(\mathrm{End}(\mathbf{1}),\mathrm{id})\to\pi_1(\mathrm{End}(\mathbf{1}),\alpha^n)$ induced by $\mathrm{End}(\mathbf{1})\to\mathrm{End}(\mathbf{1})$, $\beta\mapsto\alpha^n\circ\beta$. In particular, if σ_n vanishes in $\pi_1(\mathrm{End}(\mathbf{1}),\alpha^n)$, then α is n-symmetric in $\mathcal{C}_{/\mathbf{1}}$.

Lemma 3.6. *Let $\mathcal{C}$ be a symmetric monoidal ∞-category and let $\alpha\colon x\to\mathbf{1}$ be a symmetric object in $\mathcal{C}_{/\mathbf{1}}$. Let $X_\bullet$ be the tower $\mathbb{N}^{\mathrm{op}}\to\mathcal{C}_{/\mathbf{1}}$, $k\mapsto x^{\otimes k}$, with transition maps $\mathrm{id}\otimes\alpha$. Then the transformations*

$$\alpha\otimes\mathrm{id},\ \mathrm{id}\otimes\alpha\colon X_{\bullet+1}\to X_\bullet$$

are homotopic as maps in $\mathrm{Pro}(\mathcal{C}_{/\mathbf{1}})$.

Proof. Let σ_k be the cyclic permutation of $x^{\otimes k}$ that moves the first factor to the end. The map $\mathrm{id}\otimes\alpha\colon x^{\otimes k+1}\to x^{\otimes k}$ is then the composite of σ_{k+1}^{-1} and $\alpha\otimes\mathrm{id}$. Define a new tower $\tilde{X}_\bullet\colon\mathbb{N}^{\mathrm{op}}\to\mathcal{C}$ with $\tilde{X}_k=x^{\otimes k}$ and with transition maps $\sigma_k^{-1}\circ(\mathrm{id}\otimes\alpha)\circ\sigma_{k+1}\colon x^{\otimes k+1}\to x^{\otimes k}$. The permutations σ_k assemble into a natural equivalence $\sigma\colon\tilde{X}_\bullet\to X_\bullet$ such that $(\mathrm{id}\otimes\alpha)\circ\sigma\simeq\alpha\otimes\mathrm{id}$. The strategy of the proof is the following: we will construct an equivalence of pro-objects $\zeta\colon X_{\bullet+1}\to\tilde{X}_{\bullet+1}$ making the diagram

$$\begin{tikzcd}
X_{\bullet+1} \arrow[r, "\zeta"] \arrow[dr, "\alpha\otimes\mathrm{id}"'] & \tilde{X}_{\bullet+1} \arrow[r, "\sigma"] \arrow[d, "\alpha\otimes\mathrm{id}"] & X_{\bullet+1} \arrow[dl, "\mathrm{id}\otimes\alpha"] \\
& X_\bullet &
\end{tikzcd}$$

commute and such that $\sigma\circ\zeta$ is homotopic to the identity. Let us call π and $\tilde{\pi}$ the morphisms $\alpha\otimes\mathrm{id}\colon X_{\bullet+1}\to X_\bullet$ and $\alpha\otimes\mathrm{id}\colon\tilde{X}_{\bullet+1}\to X_\bullet$.

Suppose that α is $(n+1)$-symmetric, and let L be the 1-skeleton of the nerve of the poset $n\mathbb{N}\subset\mathbb{N}$. We will then construct ζ as a morphism in $\mathrm{Fun}(L^{\mathrm{op}},\mathcal{C})$, and we will prove that $\tilde{\pi}\circ\zeta\simeq\pi$ and $\sigma\circ\zeta\simeq\mathrm{id}$ in $\mathrm{Fun}(L^{\mathrm{op}},\mathcal{C})$. The image of an edge of L by either π or $\tilde{\pi}$ has the form

$$\begin{tikzcd}
x^{\otimes nk+1} \arrow[r] \arrow[d, "\alpha\otimes\mathrm{id}"'] \arrow[dr, "\alpha\otimes\mathrm{id}\otimes\alpha^n" description] & x^{\otimes n(k-1)+1} \arrow[d, "\alpha\otimes\mathrm{id}"] \\
x^{\otimes nk} \arrow[r, "\mathrm{id}\otimes\alpha^n"'] & x^{\otimes n(k-1)},
\end{tikzcd}$$

but π and $\tilde{\pi}$ differ on the upper triangle. Let $\sigma'_k\colon x^{\otimes nk+1}\to x^{\otimes nk+1}$ be the cyclic permutation σ_{n+1} applied to the $n+1$ factors of $x^{\otimes nk+1}$ that are killed by the diagonal. Observe that

$$\sigma_{nk+1}=(\sigma_{n(k-1)+1}\otimes\mathrm{id})\circ\sigma'_k. \tag{3.7}$$

In particular, the transition map $x^{\otimes nk+1}\to x^{\otimes n(k-1)+1}$ in $\tilde{X}_{\bullet+1}$ is $(\mathrm{id}\otimes\alpha^n)\circ\sigma'_k$. We define $\zeta\colon X_{\bullet+1}\to\tilde{X}_{\bullet+1}$ to be the identity on each vertex of L and the given homotopy $\sigma'\simeq\mathrm{id}$ on each edge. Thus, the image by ζ of an edge of L is the square

$$\begin{tikzcd}
\cdot \arrow[rr, "\mathrm{id}\otimes\alpha^n"] \arrow[d, dash] \arrow[dr, dash] & & \cdot \arrow[d, dash] \\
\cdot \arrow[r, "\sigma'"'] & \cdot \arrow[r, "\mathrm{id}\otimes\alpha^n"'] & \cdot
\end{tikzcd}$$

where untipped lines represent identity morphisms and the triangle is the given homotopy $\sigma' \simeq \mathrm{id}$. The composites $\tilde{\pi} \circ \zeta$ and $\sigma \circ \zeta$ are then described by the following pictures:

$$
\begin{tikzcd}
\cdot \arrow[rr, "\mathrm{id}\otimes\alpha^n"] \arrow[d, dash] \arrow[dr, dash] & & \cdot \arrow[d, dash] \\
\cdot \arrow[r, "\sigma'"] \arrow[d, "\alpha\otimes\mathrm{id}"'] \arrow[drr] & \cdot \arrow[r] \arrow[dr] & \cdot \arrow[d, "\alpha\otimes\mathrm{id}"] \\
\cdot \arrow[rr, "\mathrm{id}\otimes\alpha^n"'] & & \cdot
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\cdot \arrow[rr, "\mathrm{id}\otimes\alpha^n"] \arrow[d, dash] \arrow[dr, dash] & & \cdot \arrow[d, dash] \\
\cdot \arrow[r, "\sigma'"] \arrow[d, "\sigma'"'] & \cdot \arrow[r] \arrow[dl, dash] & \cdot \arrow[dd, "\sigma"] \\
\cdot \arrow[d, "\sigma\otimes\mathrm{id}"'] & & \\
\cdot \arrow[rr, "\mathrm{id}\otimes\alpha^n"'] & & \cdot
\end{tikzcd}
$$

In the first picture, the two diagonal arrows are $\alpha \otimes \mathrm{id} \otimes \alpha^n$. The assumption that the given homotopy $\sigma_{n+1} \simeq \mathrm{id}$ is a homotopy over $\mathbf{1}$ implies that the triangle with median σ' is homotopic rel its boundary to an identity 2-cell, showing that $\tilde{\pi} \circ \zeta \simeq \pi$.

Using (3.7), we inductively construct homotopies $\sigma_{nk+1} \simeq \mathrm{id}$ for $k \geq 0$. The pentagon in the second picture is the tensor product

$$
\left(
\begin{tikzcd}
x^{\otimes n(k-1)+1} \arrow[r, "\mathrm{id}"] \arrow[d, "\sigma_{n(k-1)+1}"'] & x^{\otimes n(k-1)+1} \arrow[d, "\sigma_{n(k-1)+1}"] \\
x^{\otimes n(k-1)+1} \arrow[r, "\mathrm{id}"'] & x^{\otimes n(k-1)+1}
\end{tikzcd}
\right)
\otimes
\left(
\begin{tikzcd}
x^{\otimes n} \arrow[r, "\alpha^n"] \arrow[d, "\mathrm{id}"'] & \mathbf{1} \arrow[d, "\mathrm{id}"] \\
x^{\otimes n} \arrow[r, "\alpha^n"'] & \mathbf{1}
\end{tikzcd}
\right).
$$

Using the homotopies $\sigma_{n(k-1)+1} \simeq \mathrm{id}$ and $\sigma'_k \simeq \mathrm{id}$, we obtain for every edge $e\colon \Delta^1 \to L$ a homotopy in $\mathrm{Fun}(\Delta^1, \mathcal{C})$ between $(\sigma \circ \zeta)_e$ and the identity. By construction, these homotopies agree on the common vertex of two consecutive edges of L, and hence they define a homotopy $\sigma \circ \zeta \simeq \mathrm{id}$, as desired. $\square$

Theorem 3.8. *Let $\mathcal{C}$ be a presentably symmetric monoidal ∞-category and S a set of symmetric objects of $\mathcal{C}_{/\mathbf{1}}$. Then $P_S \simeq Q_S$. More precisely, for every $E \in \mathcal{C}$, the canonical map $E \to Q_S E$ exhibits $Q_S E$ as the S-periodization of E.*

Proof. By Lemma 3.3, it suffices to show that $Q_S E$ is S-periodic, i.e., that $c_S Q_S E$ is an S_0-spectrum. We use the following explicit description of the spectrification functor Q, from the proof of [Lur09, Lemma 7.3.2.3]. Choose a regular cardinal κ such that $\mathrm{Hom}(x, -)$ preserves κ-filtered colimits for all $x \in S_0$, and choose a bijection $f\colon S_0 \to \lambda$ for some ordinal λ. Then $Q = \mathrm{colim}_{\mu < \lambda\kappa} F_\mu$, where $F_{\mu+1} = \mathrm{sh}_x^\kappa F_\mu$ if $\mu = \lambda\nu + f(x)$.

Note that each S_0-prespectrum $F_\mu(c_S E)$ is "constant" in the sense that all its terms and structure maps in a given direction are the same. For any $\alpha\colon x \to \mathbf{1}$ and $\beta\colon y \to \mathbf{1}$ in S with $\alpha \neq \beta$, it is clear that the structure map of $\mathrm{sh}_x(c_S E)$ in the y-direction is β^*. Lemma 3.6 shows that the structure map of $\mathrm{sh}_x^\omega(c_S E)$ in the x-direction is naturally homotopic to α^* under E. Hence, we have an equivalence $\mathrm{sh}_x^\omega(c_S E) \simeq c_S \Omega^\infty \mathrm{sh}_x^\omega(c_S E)$ under $c_S E$. By a straightforward transfinite induction, we can identify the towers $\{F_\mu(c_S E)\}_{\mu \leq \lambda\kappa}$ and $\{c_S \Omega^\infty F_\mu(c_S E)\}_{\mu \leq \lambda\kappa}$. In particular, $Q(c_S E) \simeq c_S Q_S E$ and $c_S Q_S E$ is an S_0-spectrum. $\square$

4. Homotopy K-theory of tame quotient stacks

The homotopy K-theory spectrum $KH(X)$ of a qcqs scheme X is the geometric realization of the simplicial spectrum $K^B(\Delta^\bullet \times X)$, where K^B is the Bass–Thomason–Trobaugh K-theory. Equivalently,

$$KH = L_{\mathbb{A}^1} K^B,$$

where $L_{\mathbb{A}^1}$ is the reflection onto the subcategory of $\mathbb{A}^1$-homotopy invariant presheaves (often called the naive $\mathbb{A}^1$-localization). There is an alternative point of view on KH due to Cisinski [Cis13]. An important feature of the Bass construction is that K^B is a Nisnevich sheaf, whereas K is not. It is also clear that the naive $\mathbb{A}^1$-localization functor $L_{\mathbb{A}^1}$ preserves Nisnevich sheaves of spectra, so that KH is not only $\mathbb{A}^1$-invariant but is also a Nisnevich sheaf. The canonical map $K \to KH$ therefore factors through the so-called motivic localization $L_{\mathrm{mot}}(K) = L_{\mathbb{A}^1} L_{\mathrm{Nis}}(K)$. But the resulting map $L_{\mathrm{mot}}(K) \to KH$ is not yet an equivalence: instead, it exhibits KH as the periodization of $L_{\mathrm{mot}}(K)$ with respect to the Bott element $\beta \in \tilde{K}_1(\mathbb{G}_m, 1)$. Our definition of the homotopy K-theory of a stack $\mathfrak{X}$ is directly analogous to this construction. The main difficulty is that we now have to deal with several Bott elements: one for each vector bundle over $\mathfrak{X}$. We also have to replace $L_{\mathbb{A}^1}$ by the more complicated homotopy localization L_{htp} [Hoy17, §3.2], which, unlike $L_{\mathbb{A}^1}$, need not preserve Nisnevich sheaves of spectra. Nevertheless, we will see that the identity $KH = L_{\mathbb{A}^1}K^B$ still holds for quotient stacks $[X/G]$ with G finite or diagonalizable.

For $\mathfrak{X} \in \mathrm{tqStk}_B$, we will denote by $K_{\mathfrak{X}}$ and $K^B_{\mathfrak{X}}$ the restrictions of K and K^B to $\mathrm{Sch}_{\mathfrak{X}}$. Let $\mathcal{E}$ be a locally free module of finite rank r over $\mathfrak{X}$,[3] $\mathbb{P}(\mathcal{E})$ the associated projective bundle, and $\mathcal{O}(1)$ the universal sheaf on $\mathbb{P}(\mathcal{E})$. By the projective bundle formula [KR18, Theorem 3.6], the functors[4]

$$\mathrm{Perf}(\mathfrak{Y}) \to \mathrm{Perf}(\mathfrak{Y} \times_{\mathfrak{X}} \mathbb{P}(\mathcal{E})), \quad E \mapsto E \boxtimes \mathcal{O}(-i), \tag{4.1}$$

for $\mathfrak{Y} \in \mathrm{Sch}_{\mathfrak{X}}$ and $0 \leq i \leq r-1$, induce an equivalence of $K_{\mathfrak{X}}$-modules

$$\prod_{i=0}^{r-1} K_{\mathfrak{X}} \simeq \mathrm{Hom}(\mathbb{P}(\mathcal{E})_+, K_{\mathfrak{X}}).$$

Let $\mathbb{V}^+(\mathcal{E})$ denote the quotient $\mathbb{P}(\mathcal{E} \oplus \mathcal{O}_{\mathfrak{X}})/\mathbb{P}(\mathcal{E})$, viewed as a pointed presheaf on $\mathrm{Sch}_{\mathfrak{X}}$. The right square in the following diagram is then commutative, and we get an equivalence as indicated:

$$\begin{tikzcd} \mathrm{Hom}(\mathbb{V}^+(\mathcal{E}), K_{\mathfrak{X}}) \arrow[r] \arrow[d, dashed, "\simeq"'] & \mathrm{Hom}(\mathbb{P}(\mathcal{E} \oplus \mathcal{O}_{\mathfrak{X}})_+, K_{\mathfrak{X}}) \arrow[r] & \mathrm{Hom}(\mathbb{P}(\mathcal{E})_+, K_{\mathfrak{X}}) \\ K_{\mathfrak{X}} & \prod_{i=0}^{r} K_{\mathfrak{X}} \arrow[l, "\mathrm{pr}_r"] \arrow[u, "\simeq"] & \prod_{i=0}^{r-1} K_{\mathfrak{X}}. \arrow[l] \arrow[u, "\simeq"] \end{tikzcd} \tag{4.2}$$

This equivalence is a morphism of $K_{\mathfrak{X}}$-modules and is therefore determined by a map $\beta_{\mathcal{E}} \colon \mathbb{V}^+(\mathcal{E}) \to K_{\mathfrak{X}}$. A standard representative of $\beta_{\mathcal{E}}$ by a perfect complex is given by the Koszul complex of the composition

$$\mathcal{E}_{\mathbb{P}(\mathcal{E}\oplus\mathcal{O})}(-1) \hookrightarrow (\mathcal{E} \oplus \mathcal{O})_{\mathbb{P}(\mathcal{E}\oplus\mathcal{O})}(-1) \twoheadrightarrow \mathcal{O}_{\mathbb{P}(\mathcal{E}\oplus\mathcal{O})}, \tag{4.3}$$

where the first map is the inclusion of the first summand and the second map is the tautological epimorphism, tensored with $\det(\mathcal{E})[r]^\vee$. In particular, the image of $\beta_{\mathcal{E}}$ in $K(\mathbb{P}(\mathcal{E} \oplus \mathcal{O}_{\mathfrak{X}}))$ can be written as

$$\sum_{i=0}^{r} c_{r-i}(\mathcal{E}) \boxtimes \mathcal{O}(-i), \quad \text{where} \quad c_{r-i}(\mathcal{E}) = (-1)^{r-i}(\det(\mathcal{E})^\vee \otimes \textstyle\bigwedge^i \mathcal{E}),$$

and it is trivialized in $K(\mathbb{P}(\mathcal{E}))$ via the Koszul complex of the tautological epimorphism $\mathcal{E}_{\mathbb{P}(\mathcal{E})}(-1) \twoheadrightarrow \mathcal{O}_{\mathbb{P}(\mathcal{E})}$.

Definition 4.4. A $K_{\mathfrak{X}}$-module is called *Bott periodic* if it is $\beta_{\mathcal{E}}$-periodic for every locally free module of finite rank $\mathcal{E}$ over $\mathfrak{X}$.

In the diagram (4.2), we can replace $K_{\mathfrak{X}}$ by $K^B_{\mathfrak{X}}$ [KR18, Theorem 3.12 (3)], and also by $L_{\mathbb{A}^1}K_{\mathfrak{X}}$ or $L_{\mathbb{A}^1}K^B_{\mathfrak{X}}$, as the projective bundle formula obviously persists after applying the naive $\mathbb{A}^1$-localization. As a result, all these $K_{\mathfrak{X}}$-modules are Bott periodic.

We denote by $KH_{\mathfrak{X}}$ the reflection of $K_{\mathfrak{X}}$ in the ∞-category of homotopy invariant, Nisnevich excisive, and Bott periodic $K_{\mathfrak{X}}$-modules. Since motivic localization and periodization are both compatible with the symmetric monoidal structure, $KH_{\mathfrak{X}}$ is an E_∞-algebra under $K_{\mathfrak{X}}$.

Definition 4.5. The *homotopy K-theory* of $\mathfrak{X} \in \mathrm{tqStk}_B$ is the E_∞-ring spectrum $KH(\mathfrak{X}) = KH_{\mathfrak{X}}(\mathfrak{X})$.

Property (1) of Theorem 1.3 is clear: if $\mathfrak{X}$ is regular, K-theory is already a homotopy invariant Nisnevich sheaf on $\mathrm{Sm}_{\mathfrak{X}}$ [Tho87, Theorems 2.7, 4.1, and 5.7], and both L_{mot} and periodization commute with restriction along $\mathrm{Sm}_{\mathfrak{X}} \hookrightarrow \mathrm{Sch}_{\mathfrak{X}}$ (since right Kan extension preserves the corresponding local objects).

For any N-quasi-projective morphism $f \colon \mathfrak{Y} \to \mathfrak{X}$, let $f^*(KH_{\mathfrak{X}})$ denote the restriction of $KH_{\mathfrak{X}}$ to $\mathrm{Sch}_{\mathfrak{Y}}$. Then $f^*(KH_{\mathfrak{X}})$ is the reflection of $K_{\mathfrak{Y}}$ in the ∞-category of $K_{\mathfrak{Y}}$-modules that are homotopy invariant, Nisnevich excisive, and periodic with respect to the maps $\beta_{f^*(\mathcal{E})}$, where $\mathcal{E}$ is a locally free module over $\mathfrak{X}$. In particular, there is a canonical morphism of E_∞-algebras $f^*(KH_{\mathfrak{X}}) \to KH_{\mathfrak{Y}}$.

[3]We do not assume that $\mathcal{E}$ has constant rank, so r is a locally constant integer on $\mathfrak{X}$. Formulas involving r must be interpreted accordingly.

[4]Here, $\mathrm{Perf}(\mathfrak{Y})$ is the stable ∞-category of perfect complexes over $\mathfrak{Y}$, i.e., dualizable objects in $\mathrm{QCoh}(\mathfrak{Y})$.

Proposition 4.6. *Let $f\colon \mathfrak{Y} \to \mathfrak{X}$ be an N-quasi-projective morphism in* tqStk_B*. Then the map $f^*(KH_{\mathfrak{X}}) \to KH_{\mathfrak{Y}}$ is an equivalence. In other words, $KH_{\mathfrak{X}}$ is the restriction of KH to* $\mathrm{Sch}_{\mathfrak{X}}$*.*

Proposition 4.6 immediately implies properties (2) and (3) of Theorem 1.3, and also that KH is a Nisnevich sheaf. Before proving it, we relate the periodization process in the definition of $KH_{\mathfrak{X}}$ to the Bass construction, which will also lead to a proof of property (4).

Consider the map $\beta_{\mathcal{O}}\colon \mathbb{P}^1/\infty \to K_{\mathfrak{X}}$. The image of $\beta_{\mathcal{O}}$ in $K_0(\mathbb{P}^1)$ is thus $[\mathcal{O}(-1)] - [\mathcal{O}]$. As an element of $\tilde{K}_0(\mathbb{P}^1, \infty)$, $\beta_{\mathcal{O}}$ is determined by any choice of trivialization of $\mathcal{O}(-1)$ over ∞; we choose the point $(1,0)$ in the line $[1:0]$. This trivialization extends to the standard trivialization of $\mathcal{O}(-1)$ over $\mathbb{P}^1 \smallsetminus 0$, which defines a lift of $\beta_{\mathcal{O}}$ to $\mathbb{P}^1/(\mathbb{P}^1 \smallsetminus 0)$. Moreover, the restriction of the latter to $\mathbb{A}^1 = \mathbb{P}^1 \smallsetminus \infty \subset \mathbb{P}^1$ is nullhomotopic via the standard trivialization of $\mathcal{O}(-1)$ over $\mathbb{P}^1 \smallsetminus \infty$. Since these two trivializations coincide over $1 \in \mathbb{G}_m$, we obtain the following commutative diagram of pointed presheaves:

$$\begin{tikzcd}
\mathbb{A}^1/\mathbb{G}_m \arrow[r, hook] \arrow[d] & \mathbb{P}^1/(\mathbb{P}^1 \smallsetminus 0) \arrow[dr, dashed] & \mathbb{P}^1/\infty \arrow[l, two heads] \arrow[d, "\beta_{\mathcal{O}}"] \\
\Sigma(\mathbb{G}_m/1) \arrow[rr, dashed, "\beta"'] & & K_{\mathfrak{X}}.
\end{tikzcd} \tag{4.7}$$

The homotopy class of the lower map is the Bott element $\beta \in \tilde{K}_1(\mathbb{G}_m, 1)$ of Example 3.4 (this identification depends on a choice of orientation of the loop in $\Sigma(\mathbb{G}_m/1)$: if the left vertical arrow in (4.7) is $* \sqcup_{\mathbb{G}_m} \mathbb{A}^1 \to * \sqcup_{\mathbb{G}_m} *$, we let the loop go from the first to the second vertex). Recall from Example 3.4 that γ is the composition of the collapse map $(\mathbb{P}^1 \smallsetminus 0) \coprod_{\mathbb{G}_m} \mathbb{A}^1 \to \Sigma(\mathbb{G}_m/1)$ and β.

Lemma 4.8. *Let E be a $K_{\mathfrak{X}}$-module.*

1. *Suppose that E is $\mathbb{A}^1$-invariant. Then E is β-periodic if and only if it is γ-periodic.*
2. *Suppose that E is a Zariski sheaf. Then E is $\beta_{\mathcal{O}}$-periodic if and only if it is γ-periodic.*

Proof. Assertion (1) follows from the fact that $(\mathbb{P}^1 \smallsetminus 0) \coprod_{\mathbb{G}_m} \mathbb{A}^1 \to \Sigma(\mathbb{G}_m/1)$ is an $L_{\mathbb{A}^1}$-equivalence. By (4.7), we can identify γ with the composition

$$(\mathbb{P}^1 \smallsetminus 0) \coprod_{\mathbb{G}_m} \mathbb{A}^1 \to \mathbb{P}^1/1 \twoheadrightarrow \mathbb{P}^1/(\mathbb{P}^1 \smallsetminus 0) \to K_{\mathfrak{X}},$$

where the first map is a Zariski equivalence. Let $\phi\colon \mathbb{P}^1/1 \to \mathbb{P}^1/\infty$ be the linear automorphism of $\mathbb{P}^1$ that fixes 0 and exchanges 1 and ∞. Then the square

$$\begin{tikzcd}
\mathbb{P}^1/1 \arrow[r, "\phi"] \arrow[d, two heads] & \mathbb{P}^1/\infty \arrow[d, "\beta_{\mathcal{O}}"] \\
\mathbb{P}^1/(\mathbb{P}^1 \smallsetminus 0) \arrow[r] & K_{\mathfrak{X}}
\end{tikzcd}$$

commutes up to homotopy, since both compositions classify the same element in $\tilde{K}_0(\mathbb{P}^1, 1)$. Assertion (2) follows. ☐

Recall from Example 3.4 that $K^B_{\mathfrak{X}} = P_\gamma K_{\mathfrak{X}}$. It follows from Lemma 4.8 that $KH_{\mathfrak{X}}$ is γ-periodic (as well as β-periodic). Hence, we have morphisms of E_∞-algebras

$$K_{\mathfrak{X}} \to K^B_{\mathfrak{X}} \to L_{\mathbb{A}^1} K^B_{\mathfrak{X}} \to KH_{\mathfrak{X}}.$$

By [KR18, Theorem 3.12], $K^B_{\mathfrak{X}}$ is Nisnevich excisive and Bott periodic, so $K^B_{\mathfrak{X}}$ is in fact the reflection of $K_{\mathfrak{X}}$ in the subcategory of Nisnevich excisive Bott periodic $K_{\mathfrak{X}}$-modules. Similarly, $L_{\mathbb{A}^1} K^B_{\mathfrak{X}}$ is the reflection of $K_{\mathfrak{X}}$ in the subcategory of $\mathbb{A}^1$-invariant, Nisnevich excisive, and Bott periodic $K_{\mathfrak{X}}$-modules. If $\mathfrak{X} \in \mathrm{Sch}_{\mathbf{B}G}$ where G is an extension of a finite group scheme by a Nisnevich-locally diagonalizable group scheme, every $\mathbb{A}^1$-invariant Nisnevich sheaf on $\mathrm{Sch}_{\mathfrak{X}}$ is already homotopy invariant [Hoy17, Remark 3.13], and so the map $L_{\mathbb{A}^1} K^B_{\mathfrak{X}} \to KH_{\mathfrak{X}}$ is an equivalence. This proves property (4) of Theorem 1.3.

We observe that the assignment $\mathcal{E} \mapsto \beta_{\mathcal{E}}$ is a functor from the groupoid of locally free modules of finite rank over $\mathfrak{X}$ to the overcategory of $K_{\mathfrak{X}}$. This functoriality comes from (4.1) and the fact that the sheaf $\mathcal{O}(-i)$

on $\mathbb{P}(\mathcal{E})$, as $\mathcal{E}$ varies in this groupoid, is a cartesian section of the fibered category of quasi-coherent sheaves. In particular, $\beta_{\mathcal{E}}\colon \mathbb{V}^+(\mathcal{E}) \to K_{\mathfrak{X}}$ coequalizes the action of linear automorphisms of $\mathcal{E}$ on $\mathbb{V}^+(\mathcal{E})$.

Write $\mathbb{V}_0(\mathcal{E})$ and $\mathbb{V}_0^+(\mathcal{E})$ for the pointed presheaves $\mathbb{V}(\mathcal{E})/(\mathbb{V}(\mathcal{E}) \smallsetminus 0)$ and $\mathbb{P}(\mathcal{E} \oplus \mathcal{O}_{\mathfrak{X}})/(\mathbb{P}(\mathcal{E} \oplus \mathcal{O}_{\mathfrak{X}}) \smallsetminus 0)$ on $\mathrm{Sch}_{\mathfrak{X}}$. As in (4.7), we have a zig-zag

$$\mathbb{V}_0(\mathcal{E}) \hookrightarrow \mathbb{V}_0^+(\mathcal{E}) \twoheadleftarrow \mathbb{V}^+(\mathcal{E}),$$

where the first map is a Zariski equivalence and the second map is an $L_{\mathbb{A}^1}$-equivalence. The map $\beta_{\mathcal{E}}\colon \mathbb{V}^+(\mathcal{E}) \to K_{\mathfrak{X}}$ extends to $\mathbb{V}_0^+(\mathcal{E})$ because the morphism (4.3) is an epimorphism away from the zero section, and hence it induces

$$\beta'_{\mathcal{E}}\colon \mathbb{V}_0(\mathcal{E}) \to K_{\mathfrak{X}}.$$

Explicitly, $\beta'_{\mathcal{E}}$ is represented by the Koszul complex of the tautological morphism $\mathcal{E}_{\mathbb{V}(\mathcal{E})} \to \mathcal{O}_{\mathbb{V}(\mathcal{E})}$ tensored with $\det(\mathcal{E})[r]^\vee$, viewed as an object of $\mathrm{Perf}(\mathbb{V}(\mathcal{E}) \text{ on } \mathfrak{X})$.

Note that the assignment $\mathcal{E} \mapsto \mathbb{V}_0(\mathcal{E})$ is right-lax symmetric monoidal, with the monoidal structure maps $\mathbb{V}_0(\mathcal{E}) \otimes \mathbb{V}_0(\mathcal{F}) \to \mathbb{V}_0(\mathcal{E} \oplus \mathcal{F})$ being Zariski equivalences. Using the Koszul complex representative of $\beta'_{\mathcal{E}}$ and the multiplicative properties of Koszul complexes, we can promote the assignment $\mathcal{E} \mapsto \beta'_{\mathcal{E}}$ to a right-lax symmetric monoidal functor from the groupoid of locally free modules of finite rank over $\mathfrak{X}$ (under direct sum) to the ∞-category of presheaves of spectra on $\mathrm{Sch}_{\mathfrak{X}}$ over $K_{\mathfrak{X}}$. In particular, if $\mathcal{E}$ and $\mathcal{F}$ are locally free modules of finite rank over $\mathfrak{X}$, we have a commutative square

$$\begin{tikzcd}
\mathbb{V}_0(\mathcal{E}) \otimes \mathbb{V}_0(\mathcal{F}) \arrow[r, "\beta'_{\mathcal{E}} \otimes \beta'_{\mathcal{F}}"] \arrow[d] & K_{\mathfrak{X}} \otimes K_{\mathfrak{X}} \arrow[d] \\
\mathbb{V}_0(\mathcal{E} \oplus \mathcal{F}) \arrow[r, "\beta'_{\mathcal{E} \oplus \mathcal{F}}"] & K_{\mathfrak{X}},
\end{tikzcd}$$

where the right vertical map is multiplication.

Proof of Proposition 4.6. We must show that the map $f^*(KH_{\mathfrak{X}}) \to \mathrm{Hom}(\mathbb{V}^+(\mathcal{E}), f^*(KH_{\mathfrak{X}}))$ induced by $\beta_{\mathcal{E}}$ is an equivalence for every locally free module $\mathcal{E}$ over $\mathfrak{Y}$. Since $f^*(KH_{\mathfrak{X}})$ is a homotopy invariant Nisnevich sheaf, we can assume that $\mathfrak{Y} \to \mathfrak{X}$ is quasi-affine [Hoy17, Proposition 4.6]. By Lemma 2.4, we can then write $\mathcal{E}$ as a quotient of $f^*(\mathcal{G})$ for some locally free module of finite rank $\mathcal{G}$ over $\mathfrak{X}$. Replacing $\mathfrak{Y}$ by an appropriate vector bundle torsor, we can assume that $f^*(\mathcal{G}) \simeq \mathcal{E} \oplus \mathcal{F}$ for some $\mathcal{F}$. Hence, $\beta'_{\mathcal{E}\oplus\mathcal{F}} \simeq \beta'_{\mathcal{E}}\beta'_{\mathcal{F}}$ acts invertibly on $f^*(KH_{\mathfrak{X}})$. In the sequence

$$f^*(KH_{\mathfrak{X}}) \xrightarrow{\beta'_{\mathcal{E}}} \mathrm{Hom}(\mathbb{V}_0(\mathcal{E}), f^*(KH_{\mathfrak{X}})) \xrightarrow{\beta'_{\mathcal{F}}} \mathrm{Hom}(\mathbb{V}_0(\mathcal{E} \oplus \mathcal{F}), f^*(KH_{\mathfrak{X}})) \xrightarrow{\beta'_{\mathcal{E}}} \mathrm{Hom}(\mathbb{V}_0(\mathcal{E} \oplus \mathcal{F} \oplus \mathcal{E}), f^*(KH_{\mathfrak{X}})),$$

the composites $\beta'_{\mathcal{F}}\beta'_{\mathcal{E}}$ and $\beta'_{\mathcal{E}}\beta'_{\mathcal{F}}$ are thus both equivalences. It then follows from the 2-out-of-6 property that all three maps are equivalences. □

This concludes the verification of properties (1)–(4) of Theorem 1.3. Finally, we would like to obtain a more concrete description of KH using Theorem 3.8. In the following lemma, $\mathrm{Sp}(\mathcal{P}_{\mathbb{A}^1,\mathrm{Zar}}(\mathrm{Sch}_{\mathfrak{X}}))$ denotes the ∞-category of $\mathbb{A}^1$-invariant Zariski sheaves of spectra on $\mathrm{Sch}_{\mathfrak{X}}$.

Lemma 4.9. *Let $\mathcal{E}$ be a locally free module of finite rank over $\mathfrak{X}$. Then $L_{\mathbb{A}^1,\mathrm{Zar}}\beta'_{\mathcal{E}}\colon L_{\mathbb{A}^1,\mathrm{Zar}}\mathbb{V}_0(\mathcal{E}) \to L_{\mathbb{A}^1,\mathrm{Zar}}K_{\mathfrak{X}}$, viewed as an object of $\mathrm{Sp}(\mathcal{P}_{\mathbb{A}^1,\mathrm{Zar}}(\mathrm{Sch}_{\mathfrak{X}}))_{/L_{\mathbb{A}^1,\mathrm{Zar}}K_{\mathfrak{X}}}$, is 3-symmetric.*

Proof. Since $\mathcal{E} \mapsto L_{\mathbb{A}^1,\mathrm{Zar}}\beta'_{\mathcal{E}}$ is symmetric monoidal, it suffices to show that $L_{\mathbb{A}^1}\sigma_3\colon L_{\mathbb{A}^1}\mathbb{V}_0(\mathcal{E}^3) \to L_{\mathbb{A}^1}\mathbb{V}_0(\mathcal{E}^3)$ is homotopic to the identity over $L_{\mathbb{A}^1}K_{\mathfrak{X}}$. The identity and σ_3 are both induced by matrices in $\mathrm{SL}_3(\mathbb{Z})$ acting on $\mathcal{E}^3$, and any two such matrices are $\mathbb{A}^1$-homotopic. Thus, it will suffice to prove the following statement: for any locally free module of finite rank $\mathcal{E}$ over $\mathfrak{X}$ and any automorphism ϕ of $p^*(\mathcal{E})$, where $p\colon \mathbb{A}^1 \times \mathfrak{X} \to \mathfrak{X}$ is the projection, the automorphisms of $\mathbb{V}_0(\mathcal{E})$ induced by ϕ_0 and ϕ_1 are $\mathbb{A}^1$-homotopic over $L_{\mathbb{A}^1}K_{\mathfrak{X}}$. Since $\beta'_{\mathcal{E}}$ is functorial in $\mathcal{E}$, the automorphism ϕ induces a commutative triangle

$$\begin{tikzcd}
\mathbb{V}_0(p^*(\mathcal{E})) \arrow[rr, "\phi"] \arrow[dr, "\beta'_{p^*(\mathcal{E})}"'] & & \mathbb{V}_0(p^*(\mathcal{E})) \arrow[dl, "\beta'_{p^*(\mathcal{E})}"] \\
& L_{\mathbb{A}^1}K_{\mathbb{A}^1\times\mathfrak{X}} &
\end{tikzcd}$$

of presheaves of spectra on $\mathrm{Sch}_{\mathbb{A}^1\times\mathfrak{X}}$. By adjunction, this is equivalent to a triangle

$$\begin{tikzcd} \mathbb{A}^1_+ \otimes \mathbb{V}_0(\mathcal{E}) \arrow[rr] \arrow[dr, "\beta'_{\mathcal{E}}"'] & & \mathbb{V}_0(\mathcal{E}) \arrow[dl, "\beta'_{\mathcal{E}}"] \\ & L_{\mathbb{A}^1}K_{\mathfrak{X}}, & \end{tikzcd}$$

which is an $\mathbb{A}^1$-homotopy between ϕ_0 and ϕ_1 over $L_{\mathbb{A}^1}K_{\mathfrak{X}}$, as desired. □

Proposition 4.10. *Let $\mathfrak{X} \in \mathrm{tqStk}_B$ and let E be a $K_{\mathfrak{X}}$-module. Then the canonical map $E \to Q_{\{\beta_{\mathcal{E}}\}}L_{\mathrm{mot}}E$ is the universal map to a homotopy invariant, Nisnevich excisive, and Bott periodic $K_{\mathfrak{X}}$-module. In particular,*

$$KH_{\mathfrak{X}} \simeq Q_{\{\beta_{\mathcal{E}}\}}L_{\mathrm{mot}}K_{\mathfrak{X}}.$$

Proof. Combining Lemma 4.9 and Theorem 3.8, we deduce that

$$P_{\{\beta'_{\mathcal{E}}\}}L_{\mathrm{mot}}E \simeq Q_{\{\beta'_{\mathcal{E}}\}}L_{\mathrm{mot}}E.$$

As $L_{\mathrm{mot}}E$ is in particular a Zariski sheaf, we can replace $\beta'_{\mathcal{E}}$ with $\beta_{\mathcal{E}}$ without changing either side. Hence, we have

$$P_{\{\beta_{\mathcal{E}}\}}L_{\mathrm{mot}}E \simeq Q_{\{\beta_{\mathcal{E}}\}}L_{\mathrm{mot}}E.$$

We conclude by noting that $Q_{\{\beta_{\mathcal{E}}\}}$ preserves homotopy invariant Nisnevich sheaves. □

In other words, $KH_{\mathfrak{X}}$ is the Bott spectrification of the motivic localization of $K_{\mathfrak{X}}$.

5. The equivariant motivic K-theory spectrum

In this final section, we prove that KH is a cdh sheaf on tqStk_B. By definition of the cdh topology, this is the case if and only if the restriction of KH to $\mathrm{Sch}_{\mathfrak{X}}$ is a cdh sheaf for every $\mathfrak{X} \in \mathrm{tqStk}_B$. Moreover, as we already know that KH is a Nisnevich sheaf, we can assume without loss of generality that $\mathfrak{X} = [X/G]$ with X a small G-scheme. By definition of smallness, we may as well assume that B has the G-resolution property and that $\mathfrak{X} = \mathbf{B}G$. Thus, we are now in the setting of [Hoy17, §6].

Let $\mathrm{H}_\bullet(\mathfrak{X})$ be the pointed motivic homotopy category over $\mathfrak{X} \in \mathrm{Sch}_{\mathbf{B}G}$, i.e., the ∞-category of pointed presheaves on $\mathrm{Sm}_{\mathfrak{X}}$ that are homotopy invariant and Nisnevich excisive. The stable motivic homotopy category over $\mathfrak{X}$ is by definition

$$\mathrm{SH}(\mathfrak{X}) = \mathrm{H}_\bullet(\mathfrak{X})[\mathrm{Sph}_{\mathbf{B}G}^{-1}],$$

where $\mathrm{Sph}_{\mathbf{B}G}$ is the collection of one-point compactifications $\mathbb{V}^+(\mathcal{E})$ of vector bundles over $\mathbf{B}G$ (pulled back to $\mathfrak{X}$); this forces the invertibility of the one-point compactifications of *all* vector bundles over $\mathfrak{X}$ [Hoy17, Corollary 6.7]. Let $\mathrm{Sp}(\mathrm{H}(\mathfrak{X}))$ be the ∞-category of homotopy invariant Nisnevich sheaves of spectra on $\mathrm{Sm}_{\mathfrak{X}}$, or equivalently the stabilization of $\mathrm{H}(\mathfrak{X})$. As a symmetric monoidal ∞-category, it is $\mathrm{H}_\bullet(\mathfrak{X})[(S^1)^{-1}]$. Since S^1 is invertible in $\mathrm{SH}(\mathfrak{X})$, we have

$$\mathrm{SH}(\mathfrak{X}) \simeq \mathrm{Sp}(\mathrm{H}(\mathfrak{X}))[\mathrm{Sph}_{\mathbf{B}G}^{-1}].$$

We also consider "big" variants of these ∞-categories: $\mathrm{Sp}(\underline{\mathrm{H}})$ is the ∞-category of homotopy invariant Nisnevich sheaves of spectra on $\mathrm{Sch}_{\mathbf{B}G}$, and $\underline{\mathrm{SH}} = \mathrm{Sp}(\underline{\mathrm{H}})[\mathrm{Sph}_{\mathbf{B}G}^{-1}]$. These ∞-categories have the following interpretation. Any presheaf on $\mathrm{Sch}_{\mathbf{B}G}$ can be restricted to $\mathrm{Sm}_{\mathfrak{X}}$ for every $\mathfrak{X} \in \mathrm{Sch}_{\mathbf{B}G}$; this gives rise to a section of the cocartesian fibration classified by $\mathfrak{X} \mapsto \mathcal{P}(\mathrm{Sm}_{\mathfrak{X}})$, which sends smooth morphisms to cocartesian edges. It is clear that this construction is an equivalence of ∞-categories between presheaves on $\mathrm{Sch}_{\mathbf{B}G}$ and such sections. From this we deduce that $\mathrm{Sp}(\underline{\mathrm{H}})$ and $\underline{\mathrm{SH}}$ can be identified with ∞-categories of sections of $\mathrm{Sp}(\mathrm{H}(-))$ and $\mathrm{SH}(-)$ over $\mathrm{Sch}_{\mathbf{B}G}^{\mathrm{op}}$ that are cocartesian over smooth morphisms.

In §4, we constructed the E_∞-algebra $KH_{\mathbf{B}G}$ in $\mathrm{Sp}(\underline{\mathrm{H}})$ as a Bott periodic $K_{\mathbf{B}G}$-module. By Proposition 3.2, there is a unique Bott periodic E_∞-algebra $\underline{\mathrm{KGL}}$ in $\underline{\mathrm{SH}}$ such that $\Omega^\infty\underline{\mathrm{KGL}} \simeq KH_{\mathbf{B}G}$, namely

$$\underline{\mathrm{KGL}} = P_{\{\beta_{\mathcal{E}}\}}\Sigma^\infty KH_{\mathbf{B}G}.$$

By Proposition 4.10, we can write $\underline{\mathrm{KGL}}$ more explicitly as an $\mathrm{Sph}_{\mathbf{B}G}$-spectrum in $\mathrm{Sp}(\underline{\mathrm{H}})$: it is the image, under the localization functor

$$QL_{\mathrm{mot}}\colon \mathrm{Stab}^{\mathrm{lax}}_{\mathrm{Sph}_{\mathbf{B}G}}\mathrm{Sp}(\mathcal{P}(\mathrm{Sch}_{\mathbf{B}G})) \to \mathrm{Stab}_{\mathrm{Sph}_{\mathbf{B}G}}\mathrm{Sp}(\underline{\mathrm{H}}) \simeq \underline{\mathrm{SH}},$$

of the "constant" $\mathrm{Sph}_{\mathbf{B}G}$-spectrum $c_{\{\beta_{\mathcal{E}}\}}K_{\mathbf{B}G}$.

Definition 5.1. For $\mathfrak{X} \in \mathrm{Sch}_{\mathbf{B}G}$, we denote by $\mathrm{KGL}_{\mathfrak{X}} \in \mathrm{CAlg}(\mathrm{SH}(\mathfrak{X}))$ the restriction of $\underline{\mathrm{KGL}}$ to $\mathrm{Sm}_{\mathfrak{X}}$.

By Proposition 4.6, the motivic spectrum $\mathrm{KGL}_{\mathfrak{X}}$ represents homotopy K-theory: for $\mathfrak{Y}$ a smooth N-quasi-projective $\mathfrak{X}$-stack, there is a natural equivalence

$$KH(\mathfrak{Y}) \simeq \mathrm{Map}^{\mathrm{Sp}}(\Sigma^\infty \mathfrak{Y}_+, \mathrm{KGL}_{\mathfrak{X}}),$$

where $\mathrm{Map}^{\mathrm{Sp}}$ denotes a mapping spectrum in the stable ∞-category $\mathrm{SH}(\mathfrak{X})$.

We now prove that $\mathfrak{X} \mapsto \mathrm{KGL}_{\mathfrak{X}}$ is a cocartesian section of $\mathrm{SH}(-)$ over $\mathrm{Sch}_{\mathbf{B}G}^{\mathrm{op}}$, i.e., that for every $f\colon \mathfrak{Y} \to \mathfrak{X}$ in $\mathrm{Sch}_{\mathbf{B}G}$, the restriction map

$$f^*(\mathrm{KGL}_{\mathfrak{X}}) \to \mathrm{KGL}_{\mathfrak{Y}}$$

in $\mathrm{SH}(\mathfrak{Y})$ is an equivalence. By [Hoy17, Corollary 6.25], this implies that KH is a cdh sheaf on $\mathrm{Sch}_{\mathbf{B}G}$ and concludes the proof of Theorem 1.3. Since $\underline{\mathrm{KGL}} = QL_{\mathrm{mot}}c_{\{\beta_\varepsilon\}}K_{\mathbf{B}G}$, the above restriction map is

$$f^*(QL_{\mathrm{mot}}c_{\{\beta_\varepsilon\}}(K|\mathrm{Sm}_{\mathfrak{X}})) \to QL_{\mathrm{mot}}c_{\{\beta_\varepsilon\}}(K|\mathrm{Sm}_{\mathfrak{Y}})$$

The localization functor QL_{mot} is compatible with the base change functor f^*, as f_* preserves local objects, so it will suffice to show that the restriction map

$$f^*(K|\mathrm{Sm}_{\mathfrak{X}}) \to K|\mathrm{Sm}_{\mathfrak{Y}} \tag{5.2}$$

is a motivic equivalence in $\mathrm{Sp}(\mathcal{P}(\mathrm{Sm}_{\mathfrak{Y}}))$.

Sending vector bundles over $\mathfrak{X}$ to their classes in K-theory induces a map of grouplike E_∞-spaces

$$\mathrm{Vect}(\mathfrak{X})^+ \to \Omega^\infty K(\mathfrak{X}), \tag{5.3}$$

where $\mathrm{Vect}(\mathfrak{X})$ is the E_∞-space of vector bundles over $\mathfrak{X}$ and $(-)^+$ denotes group completion. If $\mathfrak{X} = [X/G]$ with X a small affine G-scheme, it follows from [Hoy17, Lemma 2.17] that every short exact sequence of vector bundles over $\mathfrak{X}$ splits. In that case, the map (5.3) is an equivalence. By [Hoy17, Proposition 3.16 (2)], it follows that the map

$$\mathrm{Vect}^+ \to \Omega^\infty K|\mathrm{Sm}_{\mathfrak{X}}$$

is a motivic equivalence in $\mathcal{P}(\mathrm{Sm}_{\mathfrak{X}})$. Note also that the inclusion

$$\coprod_{n\geq 0} B_{\mathrm{fppf}}\mathrm{GL}_n \hookrightarrow \mathrm{Vect}$$

exhibits Vect as the Zariski sheafification of the subgroupoid of vector bundles of constant rank. By Lemma 5.5 below, it remains a Zariski equivalence after group completion. We therefore obtain a motivic equivalence

$$\left(\coprod_{n\geq 0} B_{\mathrm{fppf}}\mathrm{GL}_n\right)^+ \to \Omega^\infty K|\mathrm{Sm}_{\mathfrak{X}}. \tag{5.4}$$

Lemma 5.5. *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a colimit-preserving functor between presentable ∞-categories. Suppose that finite products distribute over colimits in $\mathcal{C}$ and $\mathcal{D}$ and that F preserves finite products. Then, for every E_∞-monoid M in $\mathcal{C}$, the canonical map $F(M)^+ \to F(M^+)$ is an equivalence.*

Proof. The assumption on $\mathcal{C}$ implies that the ∞-category $\mathrm{CMon}(\mathcal{C})$ of E_∞-monoids in $\mathcal{C}$ is presentable [Lur17, Corollary 3.2.3.5] and hence that group completion exists. Since both F and its right adjoint preserve finite products, they lift to a pair of adjoint functors between $\mathrm{CMon}(\mathcal{C})$ and $\mathrm{CMon}(\mathcal{D})$, as well as between the subcategories of grouplike objects. This immediately implies that F commutes with group completion. □

For any $f\colon \mathfrak{Y} \to \mathfrak{X}$ in $\mathrm{Sch}_{\mathbf{B}G}$, the pullback functor $f^*\colon \mathcal{P}(\mathrm{Sm}_{\mathfrak{X}}) \to \mathcal{P}(\mathrm{Sm}_{\mathfrak{Y}})$ preserves finite products and hence commutes with group completion of E_∞-monoids, by Lemma 5.5. Similarly, since $L_{\mathrm{mot}}\colon \mathcal{P}(\mathrm{Sm}_{\mathfrak{X}}) \to \mathrm{H}(\mathfrak{X})$ preserves finite products [Hoy17, Proposition 3.15], it commutes with group completion of E_∞-monoids. Hence, by (5.4) and Corollary 2.9 (with $\Gamma = \mathrm{GL}_n$), we deduce that the restriction map

$$f^*(\Omega^\infty K|\mathrm{Sm}_{\mathfrak{X}}) \to \Omega^\infty K|\mathrm{Sm}_{\mathfrak{Y}}$$

is a motivic equivalence in the ∞-category of grouplike E_∞-monoids in $\mathcal{P}(\mathrm{Sm}_{\mathfrak{Y}})$. Equivalently, (5.2) is a motivic equivalence in $\mathrm{Sp}_{\geq 0}(\mathcal{P}(\mathrm{Sm}_{\mathfrak{Y}}))$, whence in $\mathrm{Sp}(\mathcal{P}(\mathrm{Sm}_{\mathfrak{Y}}))$, as was to be shown.

Remark 5.6. If $f\colon \mathfrak{Y} \to \mathfrak{X}$ is a morphism of *schemes*, it is easy to show that the map (5.2) is a Zariski equivalence, because $B_{\mathrm{fppf}}\mathrm{GL}_n = B_{\mathrm{Zar}}\mathrm{GL}_n$ and GL_n is smooth. The proof of cdh descent in this case does not need the geometric model for the classifying space of GL_n.

Comments on Theorem 1.5. We discuss the minor modifications needed for the proof of Theorem 1.5. If X is a locally affine qcs G-schemes such that $|G|$ is invertible on X, then $[X/G]$ is a qcs tame Deligne–Mumford stack with coarse moduli scheme. By [KØ12, Corollary 3.8] and a noetherian approximation argument, nonconnective K-theory is a Nisnevich sheaf on such stacks, whence also KH (defined as the naive $\mathbb{A}^1$-localization of K^B). The projective bundle formula holds for general stacks [KR18, Theorem 3.6]. Hence, the restriction of KH to the category of smooth quasi-projective G-schemes over X is a homotopy invariant Nisnevich sheaf as well as a Bott periodic E_∞-algebra. By Proposition 3.2, it deloops uniquely to a Bott periodic E_∞-algebra $\mathrm{KGL}_{[X/G]} \in \mathrm{SH}([X/G])$. Since $[X/G]$ is Nisnevich-locally of the form $[U/G]$ with U affine, the proof of Theorem 2.7 and the above arguments go through (with some simplifications) and show that, for every G-equivariant morphism $f\colon Y \to X$ with Y a locally affine qcs G-scheme, $f^*(\mathrm{KGL}_{[X/G]}) \simeq \mathrm{KGL}_{[Y/G]}$. By [Hoy17, Remark 6.26], we conclude that KH satisfies cdh descent on the category of locally affine qcs G-schemes. □

Comments on Theorem 1.7. Because of the reductions done at the beginning of this section, we have only proved Theorem 1.7 with tqStk_B replaced by the subcategory of stacks $\mathfrak{X}$ admitting an N-quasi-projective map $\mathfrak{X} \to \mathbf{B}_U G$ for some B-scheme U such that $\mathbf{B}_U G$ has the resolution property. In fact, $\mathrm{SH}(\mathfrak{X})$ is only defined for such $\mathfrak{X}$ in [Hoy17, §6]. As indicated in *loc. cit.*, however, $\mathrm{SH}(-)$ extends uniquely, by right Kan extension, to a Nisnevich sheaf on tqStk_B. Hence, the section $\mathfrak{X} \mapsto \mathrm{KGL}_\mathfrak{X}$ constructed above also extends uniquely to a section of $\mathrm{CAlg}(\mathrm{SH}(-))$ on all of $\mathrm{tqStk}_B^{\mathrm{op}}$ that is cocartesian over N-quasi-projective morphisms, and Theorem 1.7 holds in the stated generality. □

Remark 5.7. Suppose that $\mathfrak{X} \in \mathrm{tqStk}_B$ is regular noetherian. Then the Borel–Moore homology theory on $\mathrm{Sch}_\mathfrak{X}$ represented by $\mathrm{KGL}_\mathfrak{X}$ is the K-theory of coherent sheaves. More precisely, for every quasi-projective morphism $f\colon \mathfrak{Z} \to \mathfrak{X}$, there is an equivalence of spectra

$$\mathrm{Map}^{\mathrm{Sp}}(\mathbf{1}_\mathfrak{Z}, f^!\mathrm{KGL}_\mathfrak{X}) \simeq K(\mathrm{Coh}(\mathfrak{Z})),$$

where the left-hand side is a mapping spectrum in $\mathrm{SH}(\mathfrak{Z})$. To prove this, write $f = p \circ i$ where $i\colon \mathfrak{Z} \hookrightarrow \mathfrak{Y}$ is a closed immersion and $p\colon \mathfrak{Y} \to \mathfrak{X}$ is smooth quasi-projective. By [Hoy17, Theorem 6.18 (2)] and Bott periodicity, $p^!\mathrm{KGL}_\mathfrak{X} \simeq \Sigma^{\Omega_p}\mathrm{KGL}_\mathfrak{Y} \simeq \mathrm{KGL}_\mathfrak{Y}$. Let $j\colon \mathfrak{U} \hookrightarrow \mathfrak{Y}$ be the open immersion complementary to i. By [Hoy17, Theorem 6.18 (4)], we have a fiber sequence

$$i_*i^!\mathrm{KGL}_\mathfrak{Y} \to \mathrm{KGL}_\mathfrak{Y} \to j_*j^*\mathrm{KGL}_\mathfrak{Y}$$

in $\mathrm{SH}(\mathfrak{Y})$, whence a fiber sequence of spectra

$$\mathrm{Map}^{\mathrm{Sp}}(\mathbf{1}_\mathfrak{Z}, f^!\mathrm{KGL}_\mathfrak{X}) \to \mathrm{Map}^{\mathrm{Sp}}(\mathbf{1}_\mathfrak{Y}, \mathrm{KGL}_\mathfrak{Y}) \to \mathrm{Map}^{\mathrm{Sp}}(\mathbf{1}_\mathfrak{U}, \mathrm{KGL}_\mathfrak{U}).$$

Since $\mathfrak{Y}$ and $\mathfrak{U}$ are regular, the second map is identified with the restriction $K(\mathfrak{Y}) \to K(\mathfrak{U})$, whose fiber is $K(\mathrm{Coh}(\mathfrak{Z}))$.

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