A TRIVIAL REMARK ON THE NISNEVICH TOPOLOGY

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Abstract. We observe that all existing definitions of the Nisnevich topology are equivalent.

Let X be a quasi-compact quasi-separated scheme and let $Y \to X$ be an étale morphism which is surjective on k-points for every field k. Then there exists a sequence $\emptyset = Z_n \subset Z_{n-1} \subset \cdots \subset Z_0 = X$ of finitely presented closed subschemes such that $Y \to X$ admits a section over $Z_{i-1} \smallsetminus Z_i$ for all i. This generalizes [MV99, §3, Lemma 1.5], where X is assumed noetherian.

Here is the proof. Consider the set Φ of all closed subschemes $Z \subset X$ for which the map $Y \times_X Z \to Z$ does not admit such a sequence. If Z is a cofiltered intersection $\bigcap_\alpha Z_\alpha$ and $Z \notin \Phi$, then there exists α such that $Z_\alpha \notin \Phi$, by [Gro66, Proposition 8.6.3 and Théorème 8.8.2 (i)]. In particular, Φ is inductively ordered. Next, we note that any étale morphism that splits over a maximal point x splits over some open neighborhood of x: indeed, the local ring $\mathcal{O}_x$ is henselian, since its reduction is a field. If $Z \in \Phi$, then Z is nonempty and hence has a maximal point. Thus, $Y \times_X Z \to Z$ splits over some nonempty open subscheme of Z, which may be chosen to have a finitely presented closed complement $W \subset Z$, by [TT90, Lemma 2.6.1 (c)]. Clearly, $W \in \Phi$. This shows that Φ does not have a minimal element. By Zorn's lemma, therefore, Φ is empty.

Consider the topology on the category of schemes generated by families of étale maps that are jointly surjective on k-points for every field k. The previous observation shows that this topology is generated by open covers and étale maps admitting finitely presented splitting sequences. In other words, the topology originally defined by Nisnevich in [Nis89] coincides with the Nisnevich topology defined in [Hoy14, Appendix C], despite an unsubstantiated claim to the contrary in *loc. cit.* By [AHW15, Proposition 2.3.2], this topology is even generated by open covers and maps of the form $\mathrm{Spec}(B \times A_f) \to \mathrm{Spec}(A)$, where $A \to B$ is étale and induces an isomorphism $A/f \simeq B/f$.

References

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Date: January 30, 2016.