FINITE BROWN REPRESENTABILITY

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In this note, we review a variant of the Brown representability theorem due to Adams, which characterizes those group-valued contravariant functors $(\mathrm{An}_*^{\mathrm{fin},\geq 1})^{\mathrm{op}} \to \mathrm{Grp}$ on *finite* pointed connected anima that are isomorphic to $\pi_0\mathrm{Map}(-, X)$ for some pointed connected anima $X \in \mathrm{An}_*^{\geq 1}$. A stable version of this result characterizes those functors $\mathrm{Sp} \to \mathrm{Ab}$ that are isomorphic to $\pi_0(- \otimes E)$ for some spectrum $E \in \mathrm{Sp}$; this is used for example to construct Landweber exact spectra. This finite/homological version of Brown representability was originally proved in [Ada71, Theorem 1.3] and an exposition in the stable setting can be found in [Mar83, Chapter 4]. However, both treatments are somewhat imprecise in a key technical step (namely in the definition of two indexing categories, called C and $\bar{C}$ in both sources; their loose definitions should presumably be understood as the posets $\mathrm{h}_0\mathcal{C}_{X//Y} \times \mathrm{h}_0\mathcal{C}_{X//Z}$ and $\mathcal{A}$ appearing in our proof of Lemma 10 below).

We will formulate and prove a generalization of Adams' result, analogous to Lurie's formulation of Brown representability [Lur17, Theorem 1.4.1.2], but the proof is essentially Adams' original argument. To state the theorem we need a few definitions.

Definition 1. Let $\mathcal{C}$ be an ∞-category. We call $\mathcal{C}$ *countable* if it has countably many isomorphism classes of objects and all the homotopy groups of all the mapping anima in $\mathcal{C}$ are countable.

Definition 2. Let $\mathcal{X}$ be an ∞-topos and let $n \geq -1$. A square

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

in $\mathcal{X}$ is called *n-cartesian* if the canonical map $A \to B \times_D C$ is n-connective.

Definition 3. Let $\mathcal{C}$ be an ∞-category with finite limits. A functor $\mathcal{C} \to \mathrm{Set}$ is called *weakly left exact* if it preserves finite products and sends cartesian squares to 0-cartesian squares. We let $\mathrm{Fun}_{\mathrm{wlex}}(\mathcal{C}, \mathrm{Set}) \subset \mathrm{Fun}(\mathcal{C}, \mathrm{Set})$ denote the full subcategory of weakly left exact functors.

Definition 4. We say that an ∞-category $\mathcal{C}$ is *generated by h-cogroups under finite colimits* if it admits all finite colimits and contains a small collection of objects $(S_\alpha)_{\alpha\in A}$ with the following properties:

(1) Each S_α admits a structure of cogroup object in the homotopy category $\mathrm{h}\mathcal{C}$.
(2) $\mathcal{C}$ is generated by $(S_\alpha)_{\alpha\in A}$ under finite colimits and retracts.

The prototypical example is the ∞-category $\mathrm{An}_*^{\mathrm{fin},\geq 1}$ of finite pointed connected anima, which is generated under finite colimits by the single cogroup object S^1. Moreover, any small stable ∞-category is generated by cogroups under finite colimits, since every object is a cogroup.

We will write $[X, Y]$ as a shorthand for $\pi_0\mathrm{Map}(X, Y)$. For comparison, we first state the usual Brown representability theorem:

Theorem 5 (Brown representability)**.** *Let $\mathcal{C}$ be an ∞-category generated by h-cogroups under finite colimits. Let $\mathrm{Fun}^{\Pi}_{\mathrm{wlex}}(\mathrm{Ind}(\mathcal{C})^{\mathrm{op}}, \mathrm{Set})$ be the full subcategory of weakly left exact functors that preserve arbitrary products. Then the functor*

$$\mathrm{hInd}(\mathcal{C}) \to \mathrm{Fun}^{\Pi}_{\mathrm{wlex}}(\mathrm{Ind}(\mathcal{C})^{\mathrm{op}}, \mathrm{Set}), \quad B \mapsto [-, B],$$

is an isomorphism.

Proof. The functor is fully faithful by Yoneda. The essential surjectivity is [Lur17, Theorem 1.4.1.2]. □

Lemma 6. *Let $\mathcal{C}$ be an ∞-category generated by h-cogroups under finite colimits. Then $\mathcal{C}$ contains a small collection of objects $(S_\alpha)_{\alpha\in A}$ such that the family of functors*

$$[S_\alpha, -]\colon \mathrm{Ind}(\mathcal{C}) \to \mathrm{Set}$$

is conservative.

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Proof. For a morphism $\varepsilon\colon X \to \emptyset$ in $\mathcal{C}$, denote by $\Sigma_\varepsilon X$ the pushout $\emptyset \sqcup_X \emptyset$, so that $\mathrm{Map}(\Sigma_\varepsilon X, Y) = \Omega_\varepsilon \mathrm{Map}(X,Y)$ for any $Y \in \mathrm{Ind}(\mathcal{C})$. Let $(S_\alpha)_{\alpha\in A}$ be a family of h-cogroups generating $\mathcal{C}$ under finite colimits and retracts, with counit maps $\varepsilon\colon S_\alpha \to \emptyset$. Then $(S_\alpha)_{\alpha\in A}$ generates $\mathrm{Ind}(\mathcal{C})$ under colimits, so that the family of functors $\mathrm{Map}(S_\alpha,-)\colon \mathrm{Ind}(\mathcal{C}) \to \mathrm{hAn}$ is conservative. Since S_α is a cogroup in $\mathrm{h}\mathcal{C}$, each of these functors factors through Grp(hAn). But a morphism in Grp(hAn) is an isomorphism if and only if it induces an isomorphism on all homotopy groups at the unit element. Therefore the family $(\Sigma^n_\varepsilon S_\alpha)_{\alpha\in A, n\in\mathbb{N}}$ has the desired property. □

Theorem 7 (finite Brown representability)**.** *Let $\mathcal{C}$ be a pointed ∞-category generated by h-cogroups under finite colimits and such that $\mathrm{h}\mathcal{C}$ is countable. Let $F\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Grp}$ be a weakly left exact functor and let $\hat{F}\colon \mathrm{Ind}(\mathcal{C})^{\mathrm{op}} \to \mathrm{Set}$ be the extension of F that preserves cofiltered limits. Then there exists an object $B \in \mathrm{Ind}(\mathcal{C})$ and a natural isomorphism $[-,B] \simeq F$ of functors $\mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$. Moreover, B is uniquely determined up to isomorphism, and the induced natural transformation $[-,B] \to \hat{F}$ of functors $\mathrm{Ind}(\mathcal{C})^{\mathrm{op}} \to \mathrm{Set}$ is objectwise surjective.*

Proof. The proof is exactly the same as that of [Lur17, Theorem 1.4.1.2] using Lemma 10 below, where the additional assumptions that $\mathcal{C}$ is pointed, that $\mathrm{h}\mathcal{C}$ is countable, and that F is group-valued are used. We repeat the argument for the reader's convenience.

Let $(S_\alpha)_{\alpha\in A}$ be a family of objects of $\mathcal{C}$ as in Lemma 6. We start by proving the following assertion:

(∗) Let $X \in \mathrm{Ind}(\mathcal{C})$ and let $x \in \hat{F}(X)$. Then there exists a map $X \to X'$ in $\mathrm{Ind}(\mathcal{C})$ and an element $x' \in \hat{F}(X')$ lifting x and inducing bijections $[S_\alpha, X'] \to F(S_\alpha)$ for all $\alpha \in A$.

To that end we construct a sequence $X \to X_0 \to X_1 \to X_2 \to \dots$ in $\mathrm{Ind}(\mathcal{C})$ and compatible elements $x_n \in \hat{F}(X_n)$ lifting x. Set $X_0 = X \sqcup \coprod_{\alpha\in A, s\in F(S_\alpha)} S_\alpha$. Since $\hat{F}$ takes arbitrary coproducts to products, there exists $x_0 \in \hat{F}(X_0)$ lifting x as well as all the elements $s \in F(S_\alpha)$ for all $\alpha \in A$. Thus x_0 induces a surjection $[S_\alpha, X_0] \twoheadrightarrow F(S_\alpha)$ for every $\alpha \in A$.

Suppose that (X_n, x_n) has been constructed. Let R_α be the equivalence relation on $[S_\alpha, X_n]$ such that x_n induces an injective map $[S_\alpha, X_n]/R_\alpha \hookrightarrow F(S_\alpha)$. We define X_{n+1} by the pushout square

$$\begin{array}{ccc} \coprod_{\alpha\in A, r\in R_\alpha}(S_\alpha \sqcup S_\alpha) & \xrightarrow{r} & X_n \\ {\scriptstyle\nabla}\downarrow & & \downarrow \\ \coprod_{\alpha\in A, r\in R_\alpha} S_\alpha & \longrightarrow & X_{n+1}. \end{array}$$

By Lemma 10, there exists an element $x_{n+1} \in \hat{F}(X_{n+1})$ lifting x_n.

Finally, let $X' = \mathrm{colim}_n X_n$. Then the sequence of elements x_n defines an element $x' \in \hat{F}(X')$ lifting x. The induced map $[S_\alpha, X'] \to F(S_\alpha)$ is surjective, since the composite $[S_\alpha, X_0] \to [S_\alpha, X'] \to F(S_\alpha)$ was already surjective. To prove the injectivity of this map, let $f, g\colon S_\alpha \to X'$ be such that $f^*(x') = g^*(x')$. Since S_α is compact in $\mathrm{Ind}(\mathcal{C})$, f and g factor through maps $f_n, g_n\colon S_\alpha \to X_n$ for some n, so that $f_n^*(x_n) = g_n^*(x_n)$. By construction, the composite map $S_\alpha \sqcup S_\alpha \to X_n \to X_{n+1}$ factors through $S_\alpha \sqcup S_\alpha \to S_\alpha$, whence $f = g$. This concludes the proof of (∗).

Let $B \in \mathrm{Ind}(\mathcal{C})$ and $b \in \hat{F}(B)$ satisfy (∗) for $X = \emptyset$. The element b defines a natural transformation $[-,B] \to \hat{F}$, which we claim has the desired properties. We first prove the surjectivity of the natural transformation. Let $X \in \mathrm{Ind}(\mathcal{C})$ and let $x \in \hat{F}(X)$. Applying (∗) to the element $(b,x) \in \hat{F}(B \sqcup X)$, we obtain a morphism $B \sqcup X \to X'$ and an element $x' \in \hat{F}(X')$ lifting (b,x) and inducing bijections $[S_\alpha, X'] \to F(S_\alpha)$. It follows that the map $B \to X'$ induces bijections $[S_\alpha, B] \to [S_\alpha, X']$ for all $\alpha \in A$, so that it is an isomorphism in $\mathrm{Ind}(\mathcal{C})$. The composite $X \to X' \simeq B$ is then a preimage of x, as desired.

We now prove the injectivity of the natural transformation on $\mathcal{C}$. Let $X \in \mathcal{C}$ and let $f, g\colon X \to B$ be two preimages of some element $x \in F(X)$, i.e., $f^*(b) = x = g^*(b)$. We form the pushout square

$$\begin{array}{ccc} X \sqcup X & \xrightarrow{f+g} & B \\ {\scriptstyle\nabla}\downarrow & & \downarrow \\ X & \longrightarrow & W \end{array}$$

in $\mathrm{Ind}(\mathcal{C})$. By Lemma 10, we find $w \in \hat{F}(W)$ lifting x and b. Applying (∗) to this element, we find a morphism $W \to W'$ and an element $w' \in \hat{F}(W')$ lifting w and inducing bijections $[S_\alpha, W'] \to F(S_\alpha)$. The composite map $h\colon B \to W'$ then induces bijections on $[S_\alpha, -]$ for all $\alpha \in A$, so that it is an isomorphism in $\mathrm{Ind}(\mathcal{C})$. Since $h \circ f = h \circ g$, we deduce that $f = g$, as desired.

It remains to prove the uniqueness of B up to isomorphism. Let $C \in \mathrm{Ind}(\mathcal{C})$ be any object with a natural isomorphism $[-, C] \simeq F$ on $\mathcal{C}$, and let $c \in \hat{F}(C)$ be the element whose restriction to any $X \in \mathcal{C}_{/C}$ corresponds to the homotopy class of $X \to C$. Applying $(*)$ to the pair $(b, c) \in \hat{F}(B \sqcup C)$, we find a map $B \sqcup C \to B'$ and an element $b' \in \hat{F}(B')$ lifting b and c and inducing bijections $[S_\alpha, B'] \to F(S_\alpha)$ for all $\alpha \in A$. Then both maps $B \to B'$ and $C \to B'$ induce bijections on $[S_\alpha, -]$ for all $\alpha \in A$, so that they are isomorphisms in $\mathrm{Ind}(\mathcal{C})$. □

Remark 8. The cogroup generation assumption in Theorems 5 and 7 is only used through Lemma 6. One may therefore replace it by the conclusion of Lemma 6. However, outside of 1-categories, we do not know any examples where this weaker assumption can be checked directly.

Lemma 9. *Let $\mathcal{A}$ be a countable filtered poset and $F\colon \mathcal{A}^{\mathrm{op}} \to \mathrm{Set}$ a nonempty functor sending all morphisms to surjections. Then the limit of F is nonempty.*

Proof. Since $\mathcal{A}$ is a countable filtered poset, there exists a cofinal map $\mathbb{N} \to \mathcal{A}$. We may thus assume $\mathcal{A} = \mathbb{N}$, in which case the assertion is clear. □

Lemma 10. *Let $\mathcal{C}$ be a pointed ∞-category with finite colimits such that $\mathrm{h}\mathcal{C}$ is countable, let $F\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Grp}$ be a weakly left exact functor, and let $\hat{F}\colon \mathrm{Ind}(\mathcal{C})^{\mathrm{op}} \to \mathrm{Set}$ be the extension of F that preserves cofiltered limits. Let*

$$\begin{array}{ccc} X & \longrightarrow & Z \\ \downarrow & & \downarrow \\ Y & \longrightarrow & W \end{array}$$

be a pushout square in $\mathrm{Ind}(\mathcal{C})$. If $X \simeq \coprod_{i \in I} X_i$ with $X_i \in \mathcal{C}$, then $\hat{F}$ takes this square to a 0-cartesian square.

Proof. We first prove that the lemma holds whenever $X \in \mathcal{C}$. We fix $(y, z) \in \hat{F}(Y) \times_{F(X)} \hat{F}(Z)$ and we seek $w \in \hat{F}(W)$ lifting y and z. For any factorizations $X \to Y' \to Y$ and $X \to Z' \to Z$ with $Y', Z' \in \mathcal{C}$, let $W' = Y' \sqcup_X Z'$ and let $\mathrm{Lift}_{y,z}(Y', Z') \subset F(W')$ be the set of elements lifting both $y|Y'$ and $z|Z'$. This defines a functor

$$\mathrm{Lift}_{y,z}\colon \mathcal{C}^{\mathrm{op}}_{X//Y} \times \mathcal{C}^{\mathrm{op}}_{X//Z} \to \mathrm{Set}.$$

Note that W itself is the colimit of the filtered diagram $\mathcal{C}_{X//Y} \times \mathcal{C}_{X//Z} \to \mathcal{C}$ sending (Y', Z') to W'. An element $w \in \hat{F}(W)$ lifting y and z is therefore exactly an element in the limit of the functor $\mathrm{Lift}_{y,z}$. Thus, we have to show that the limit of $\mathrm{Lift}_{y,z}$ is nonempty. To do so, it suffices to find a factorization $\mathcal{C}^{\mathrm{op}}_{X//Y} \times \mathcal{C}^{\mathrm{op}}_{X//Z} \to \mathcal{A}^{\mathrm{op}} \to \mathrm{Set}$ of $\mathrm{Lift}_{y,z}$ such that the limit of $\mathcal{A}^{\mathrm{op}} \to \mathrm{Set}$ is nonempty. We will now construct such a factorization where $\mathcal{A}$ is a filtered poset.

Let $Y' \to Y''$ and $Z' \to Z''$ be morphisms in $\mathcal{C}_{X//Y}$ and $\mathcal{C}_{X//Z}$. We consider the pushout squares

$$\begin{array}{ccccc} X & \longrightarrow & Z' & \longrightarrow & Z'' \\ \downarrow & & \downarrow & & \downarrow \\ Y' & \longrightarrow & W' & \longrightarrow & V \\ \downarrow & & \downarrow & & \downarrow \\ Y'' & \longrightarrow & U & \longrightarrow & W''. \end{array}$$

Since F takes these squares to 0-cartesian squares, the restriction map $\mathrm{Lift}_{y,z}(Y'', Z'') \to \mathrm{Lift}_{y,z}(Y', Z')$ is surjective. This implies that the functor $\mathrm{Lift}_{y,z}$ identifies parallel morphisms (since they can be coequalized in $\mathcal{C}_{X//Y} \times \mathcal{C}_{X//Z}$), and hence factors through the homotopy 0-category $\mathrm{h}_0\mathcal{C}_{X//Y} \times \mathrm{h}_0\mathcal{C}_{X//Z}$, which is a filtered poset.

We define a new 0-category $\mathcal{A}$ as follows: its objects are those of $\mathcal{C}_{X//Y} \times \mathcal{C}_{X//Z}$, and we set $(Y', Z') \leq (Y'', Z'')$ if for any morphisms $Y' \to Y''' \leftarrow Y''$ in $\mathcal{C}_{X//Y}$ and $Z' \to Z''' \leftarrow Z''$ in $\mathcal{C}_{X//Z}$, the surjection $\mathrm{Lift}_{y,z}(Y''', Z''') \twoheadrightarrow \mathrm{Lift}_{y,z}(Y', Z')$ factors through $\mathrm{Lift}_{y,z}(Y''', Z''') \twoheadrightarrow \mathrm{Lift}_{y,z}(Y'', Z'')$. The factorization $\mathrm{Lift}_{y,z}(Y'', Z'') \to \mathrm{Lift}_{y,z}(Y', Z')$ is then surjective and independent of Y''' and Z''', since $\mathcal{C}_{X//Y}$ and $\mathcal{C}_{X//Z}$ are filtered. Moreover, if there exists a morphism $(Y', Z') \to (Y'', Z'')$ in $\mathcal{C}_{X//Y} \times \mathcal{C}_{X//Z}$, then $(Y', Z') \leq (Y'', Z'')$. The functor $\mathrm{Lift}_{y,z}$ thus factors as

$$\mathcal{C}^{\mathrm{op}}_{X//Y} \times \mathcal{C}^{\mathrm{op}}_{X//Z} \to \mathrm{h}_0\mathcal{C}^{\mathrm{op}}_{X//Y} \times \mathrm{h}_0\mathcal{C}^{\mathrm{op}}_{X//Z} \to \mathcal{A}^{\mathrm{op}} \to \mathrm{Set},$$

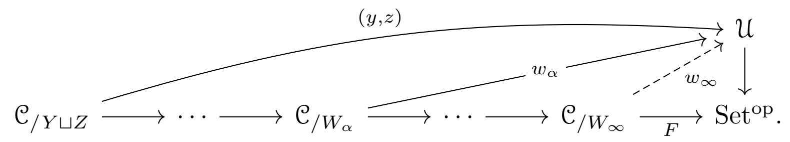
where the first two functors are essentially surjective and the last functor sends (X, X) to a point and any morphism to a surjection. To conclude, we show that the filtered poset $\mathcal{A}$ is countable, so that the limit of $\mathcal{A}^{\mathrm{op}} \to \mathrm{Set}$ is automatically nonempty by Lemma 9.

We now use for the first time the assumptions that $\mathcal{C}$ is pointed and that F is group-valued: we have an exact sequence of groups

$$F(Y' \sqcup Z') \leftarrow F(W') \leftarrow F(\Sigma X) \leftarrow F(\Sigma(Y' \sqcup Z')).$$

The set $\mathrm{Lift}_{y,z}(Y', Z')$ is thus a coset of the image of $F(\Sigma X)$ in $F(W')$, and this image is the quotient of $F(\Sigma X)$ by the image of $F(\Sigma(Y' \sqcup Z'))$. It follows that $(Y', Z') \le (Y'', Z'')$ if (and only if) the image of $F(\Sigma(Y' \sqcup Z'))$ in $F(\Sigma X)$ contains the image of $F(\Sigma(Y'' \sqcup Z''))$. Since $\mathcal{C}_{\Sigma X/}$ has countably many isomorphism classes, there are only countably many possible such images, hence only countably many isomorphism classes in $\mathcal{A}$. This completes the proof of the lemma when $X \in \mathcal{C}$.

The proof in the general case is now an easy application of Zorn's lemma. Let $(X_i)_{i \in I}$ be a family of objects of $\mathcal{C}$ and let $\coprod_{i \in I} X_i \to Y$ and $\coprod_{i \in I} X_i \to Z$ be morphisms in $\mathrm{Ind}(\mathcal{C})$. For any subset $J \subset I$, let $X_J = \coprod_{i \in J} X_i$ and let $W_J = Y \sqcup_{X_J} Z$. We fix $(y, z) \in \hat{F}(Y) \times_{\hat{F}(X_I)} \hat{F}(Z)$ and we seek $w \in \hat{F}(W_I)$ lifting y and z. Let P be the poset of pairs (J, w) where $J \subset I$ and $w \in \hat{F}(W_J)$ lifts y and z. We first show that every chain in P has an upper bound. Let A be a totally ordered set and let $A \to P$, $\alpha \mapsto (J_\alpha, w_\alpha)$. Let $J_\infty = \bigcup_{\alpha \in A} J_\alpha$, and for $\alpha \in A \cup \{\infty\}$ let $X_\alpha = X_{J_\alpha}$ and $W_\alpha = w_{J_\alpha}$. Let $\mathcal{U} \to \mathrm{Set}^{\mathrm{op}}$ be the universal cartesian fibration in sets. The given chain (J_α, w_α) can be represented by a commutative diagram



Since $W_\infty = \mathrm{colim}_{\alpha \in A} W_\alpha$ in $\mathrm{Ind}(\mathcal{C})_{Y \sqcup Z/}$, the ∞-category $\mathcal{C}_{/W_\infty}$ is the colimit under $\mathcal{C}_{/Y \sqcup Z}$ of the ∞-categories $\mathcal{C}_{/W_\alpha}$. Thus there exists an element $w_\infty \in \hat{F}(W_\infty)$ lifting y, z, and all the w_α's as indicated. Then (J_∞, w_∞) is an upper bound of the given chain in P. By Zorn's lemma, the poset P admits a maximal element $(J_{\max}, w_{\max})$. It remains to show that $J_{\max} = I$. But if $(J, w) \in P$ and $i \in I - J$, then since $W_{J \cup \{i\}} = W_J \sqcup_{X_i \sqcup X_i} X_i$ and $X_i \sqcup X_i \in \mathcal{C}$, we can lift $w \in \hat{F}(W_J)$ to $\hat{F}(W_{J \cup \{i\}})$, so (J, w) is not maximal. □

Remark 11. The conclusion of Lemma 10 does *not* hold for arbitrary $X \in \mathrm{Ind}(\mathcal{C})$. If it did, then the proof of Theorem 7 would show that the natural transformation $[-, B] \to \hat{F}$ is bijective on all of $\mathrm{Ind}(\mathcal{C})$, but this is usually not the case. For example, for a sequential ind-object, the failure of injectivity is measured by the Milnor exact sequence.

Remark 12. The natural transformation $[-, B] \to \hat{F}$ in the conclusion of Theorem 7 is more generally an isomorphism on arbitrary coproducts of objects of $\mathcal{C}$, since both sides transform coproducts into products. Without the assumptions that $\mathcal{C}$ is pointed, that F is group-valued, and that $\mathrm{h}\mathcal{C}$ is countable, the conclusion of Lemma 10 is therefore *necessary and sufficient* for the conclusion of Theorem 7. We do not know if the first two assumptions are necessary. However, we cannot remove the countability assumption in the theorem (and hence in the lemma): if $\mathcal{C}$ is a small stable ∞-category satisfying the conclusion of Theorem 7, then the functor $\mathrm{hInd}(\mathcal{C}) \to \mathrm{Ind}(\mathrm{h}\mathcal{C})$ is full. But this fails for example for the ∞-category of κ-compact spectra with κ an uncountable cardinal.

Remark 13. The assumption that $\mathrm{h}\mathcal{C}$ is countable in Lemma 10 (and hence in Theorem 7) can be weakened to the assumption that $\mathcal{C}_{\Sigma X/}$ has countably many isomorphism classes for every $X \in \mathcal{C}$. Combining this observation with Remark 8, we deduce the following somewhat curious fact (which may well be obvious for other reasons): if $\mathcal{C}$ is a pointed 1-category with finite colimits and with countably many isomorphism classes, then every weakly left exact functor $\mathcal{C}^{\mathrm{op}} \to \mathrm{Grp}$ is in fact left exact.

Corollary 14 (homological Brown representability)**.** *Let $\mathcal{C}$ be a symmetric monoidal pointed ∞-category generated by h-groups under finite limits, such that* $\mathrm{h}\mathcal{C}$ *is countable and every object of* $\mathcal{C}$ *is dualizable. Let* $F\colon \mathcal{C} \to \mathrm{Grp}$ *be a weakly left exact functor and let* $\hat{F}\colon \mathrm{Ind}(\mathcal{C}) \to \mathrm{Set}$ *be the extension of F that preserves filtered colimits. Then there exists an object $E \in \mathrm{Ind}(\mathcal{C})$ and a natural isomorphism* $[\mathbf{1}, - \otimes E] \xrightarrow{\sim} \hat{F}$*. Moreover, E is uniquely determined up to isomorphism.*

Proof. The assumption that every object of $\mathcal{C}$ is dualizable implies that $\mathcal{C}$ is self-dual, so that $\mathcal{C}$ is also generated by h-cogroups under finite colimits. Let $G\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Grp}$ be the functor given by $G(X) = F(X^\vee)$. Then G satisfies the assumptions of Theorem 7. Thus, there exists $E \in \mathrm{Ind}(\mathcal{C})$ and a natural isomorphism

$[-, E] \simeq G$ on $\mathcal{C}$, which determines E uniquely up to isomorphism. Under the duality isomorphism $\mathcal{C} \simeq \mathcal{C}^{\mathrm{op}}$, this amounts to a natural isomorphism $[\mathbf{1}, - \otimes E] \simeq F$. To conclude, we note that the functor $[\mathbf{1}, - \otimes E] \colon \mathrm{Ind}(\mathcal{C}) \to \mathrm{Set}$ preserves filtered colimits, hence is isomorphic to the extension $\hat{F}$. □

We now would like to promote Theorem 7 to an isomorphism of categories, as in the statement of Theorem 5.

Definition 15. Let $\mathcal{C}$ be an ∞-category. Two maps $f, g \colon X \to Y$ in $\mathrm{Ind}(\mathcal{C})$ are called *weakly homotopic* if for every $K \in \mathcal{C}$ and every map $h \colon K \to X$, the composites $f \circ h$ and $g \circ h$ are homotopic. We write $[X, Y]_w$ for the set of weak homotopy classes of maps $X \to Y$ in $\mathrm{Ind}(\mathcal{C})$. These are the morphisms of a 1-category $\mathrm{h}_w\mathrm{Ind}(\mathcal{C})$, which contains $\mathrm{h}\mathcal{C}$ as a full subcategory.

If $\mathcal{C}$ is additive, a map in $\mathrm{Ind}(\mathcal{C})$ is also called a *phantom map* if it is weakly homotopic to 0. In this case, $[X, Y]_w$ is the quotient of the group $[X, Y]$ by the subgroup of phantom maps.

Remark 16. The canonical functor $\mathrm{hInd}(\mathcal{C}) \to \mathrm{h}_w\mathrm{Ind}(\mathcal{C})$ is full and essentially surjective. Moreover, it preserves any limits that exist in $\mathrm{hInd}(\mathcal{C})$. If $\mathcal{C}$ is generated by h-cogroups under finite colimits, this functor is also conservative (by Lemma 6).

Lemma 17. *Let $\mathcal{C}$ be as in Theorem 7. Let $F \colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Grp}$ be a weakly left exact functor and let $B \in \mathrm{Ind}(\mathcal{C})$ be an object with a natural isomorphism $[-, B] \simeq F$. For every $X \in \mathrm{Ind}(\mathcal{C})$, the map $[X, B] \to \hat{F}(X)$ induces an isomorphism $[X, B]_w \xrightarrow{\sim} \hat{F}(X)$.*

Proof. The last two statements in Theorem 7 imply that the map $[X, B] \to \hat{F}(X)$ is surjective. Since the natural transformation $[-, B] \to \hat{F}$ is an isomorphism on $\mathcal{C}$, it is clear that two morphisms $f, g \colon X \to B$ become equal in $\hat{F}(X)$ if and only if they are weakly homotopic. □

Proposition 18 (finite/homological Brown representability for natural transformations)**.**

(1) *Let $\mathcal{C}$ be an ∞-category as in Theorem 7. Then the functor*

$$\mathrm{h}_w\mathrm{Ind}(\mathcal{C}) \to \mathrm{Fun}_{\mathrm{wlex}}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set}), \quad B \mapsto [-, B],$$

detects group objects and restricts to an isomorphism between the full subcategories of objects that admit group structures. In particular, it induces an isomorphism

$$\mathrm{Grp}(\mathrm{h}_w\mathrm{Ind}(\mathcal{C})) \xrightarrow{\sim} \mathrm{Fun}_{\mathrm{wlex}}(\mathcal{C}^{\mathrm{op}}, \mathrm{Grp}).$$

(2) *Let $\mathcal{C}$ be a symmetric monoidal ∞-category as in Corollary 14. Then the functor*

$$\mathrm{h}_w\mathrm{Ind}(\mathcal{C}) \to \mathrm{Fun}_{\mathrm{wlex}}(\mathcal{C}, \mathrm{Set}), \quad E \mapsto [\mathbf{1}, - \otimes E],$$

detects group objects and restricts to an isomorphism between the full subcategories of objects that admit group structures. In particular, it induces an isomorphism

$$\mathrm{Grp}(\mathrm{h}_w\mathrm{Ind}(\mathcal{C})) \xrightarrow{\sim} \mathrm{Fun}_{\mathrm{wlex}}(\mathcal{C}, \mathrm{Grp}).$$

Proof. (2) is a rephrasing of (1) under the duality isomorphism $\mathcal{C} \simeq \mathcal{C}^{\mathrm{op}}$. Let $\mathcal{G} \subset \mathrm{Fun}_{\mathrm{wlex}}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set})$ be the full subcategory of objects that admit group structures, and let $\mathcal{H}$ be its preimage in $\mathrm{h}_w\mathrm{Ind}(\mathcal{C})$. Since $\mathcal{G}$ is closed under finite products, it suffices to show that the functor $\mathcal{H} \to \mathcal{G}$ is an isomorphism. It is essentially surjective by Theorem 7. Let $X \in \mathrm{Ind}(\mathcal{C})$, let $(X_\alpha)_\alpha$ be a filtered diagram in $\mathcal{C}$ with colimit X, and let $F \colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$. By Yoneda, we have

$$\mathrm{Map}([-, X], F) = \mathrm{Map}(\operatorname*{colim}_\alpha [-, X_\alpha], F) = \lim_\alpha \mathrm{Map}([-, X_\alpha], F) = \lim_\alpha F(X_\alpha) = \hat{F}(X).$$

It then follows from Lemma 17 that the map

$$[X, B]_w \to \mathrm{Map}([-, X], [-, B])$$

is an isomorphism for any $B \in \mathcal{H}$. In particular, $\mathcal{H} \to \mathcal{G}$ is fully faithful. □

Finally, we specialize Proposition 18 to the additive (e.g., stable) case. Note that a small additive ∞-category $\mathcal{C}$ with finite colimits is generated by h-cogroups under finite colimits. Moreover, every weakly left exact functor $\mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ has a unique group structure.

Corollary 19. *Let $\mathcal{C}$ be an additive ∞-category with finite colimits such that $\mathrm{h}\mathcal{C}$ is countable.*

(1) *There is an isomorphism*

$$\mathrm{h}_w\mathrm{Ind}(\mathcal{C}) \xrightarrow{\sim} \mathrm{Fun}_{\mathrm{wlex}}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set}), \quad B \mapsto [-, B].$$

(2) *Suppose that* $\mathcal{C}$ *has a symmetric monoidal structure in which every object is dualizable. Then there is an isomorphism*

$$\mathrm{h}_w\mathrm{Ind}(\mathcal{C}) \xrightarrow{\sim} \mathrm{Fun}_{\mathrm{wlex}}(\mathcal{C}, \mathrm{Set}), \quad E \mapsto [\mathbf{1}, - \otimes E].$$

Let now $\mathcal{C}$ be a pointed ∞-category with finite colimits. The *Spanier–Whitehead* ∞*-category* $\mathrm{SW}(\mathcal{C})$ is the colimit of the sequence

$$\mathcal{C} \xrightarrow{\Sigma} \mathcal{C} \xrightarrow{\Sigma} \mathcal{C} \longrightarrow \cdots.$$

It is the universal stable ∞-category with a right exact functor from $\mathcal{C}$, and we have

$$\mathrm{Ind}(\mathrm{SW}(\mathcal{C})) = \mathrm{Sp}(\mathrm{Ind}(\mathcal{C})).$$

A weakly left exact functor $\mathrm{SW}(\mathcal{C})^{\mathrm{op}} \to \mathrm{Set}$ is called a *cohomology theory* on $\mathcal{C}$: it is equivalently a sequence of weakly left exact functors $H^n\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ with isomorphisms $H^n \simeq H^{n+1} \circ \Sigma$. A weakly left exact functor $\mathrm{SW}(\mathcal{C}) \to \mathrm{Set}$ is called a *homology theory* on $\mathcal{C}$: it is a sequence of functor $H_n\colon \mathcal{C} \to \mathrm{Set}$ that transform finite coproducts into finite products (this makes sense as $\mathcal{C}$ is pointed) and take pushout squares to 0-cartesian squares, with isomorphisms $H_n \simeq H_{n+1} \circ \Sigma$. We denote by $\mathrm{CohTh}(\mathcal{C})$ (resp. by $\mathrm{HomTh}(\mathcal{C})$) the category of cohomology theories (resp. of homology theories) on $\mathcal{C}$.

Since $\mathrm{hSW}(\mathcal{C})$ is countable if $\mathrm{h}\mathcal{C}$ is countable, we obtain the following special case of Corollary 19:

Corollary 20. *Let* $\mathcal{C}$ *be a pointed* ∞*-category with finite colimits such that* $\mathrm{h}\mathcal{C}$ *is countable.*

(1) *There is an isomorphism*

$$\mathrm{h}_w\mathrm{Sp}(\mathrm{Ind}(\mathcal{C})) \xrightarrow{\sim} \mathrm{CohTh}(\mathcal{C}), \quad (B_n)_{n\in\mathbb{Z}} \mapsto ([-, B_n])_{n\in\mathbb{Z}}.$$

(2) *Suppose that* $\mathcal{C}$ *has a symmetric monoidal structure that preserves finite colimits in each variable, such that every object becomes dualizable in* $\mathrm{SW}(\mathcal{C})$*. Then there is an isomorphism*

$$\mathrm{h}_w\mathrm{Sp}(\mathrm{Ind}(\mathcal{C})) \xrightarrow{\sim} \mathrm{HomTh}(\mathcal{C}), \quad E \mapsto ([\mathbf{1}, \Sigma^{\infty-n}(-) \otimes E])_{n\in\mathbb{Z}}.$$

Remark 21. The analogue of Corollary 20(1) in the setting of the standard Brown representability theorem is as follows. Let $\mathcal{C}$ be a small pointed ∞-category with finite colimits, and let $\mathrm{CohTh}^{\Pi}(\mathrm{Ind}(\mathcal{C}))$ be the full subcategory of cohomology theories $(H^n)_{n\in\mathbb{Z}}$ on $\mathrm{Ind}(\mathcal{C})$ such that each $H^n\colon \mathrm{Ind}(\mathcal{C})^{\mathrm{op}} \to \mathrm{Set}$ preserves arbitrary products. Define $\mathrm{Sp}_{\Omega}(\mathrm{hInd}(\mathcal{C}))$ to be the limit of the tower

$$\cdots \longrightarrow \mathrm{hInd}(\mathcal{C}) \xrightarrow{\Omega} \mathrm{hInd}(\mathcal{C}) \xrightarrow{\Omega} \mathrm{hInd}(\mathcal{C}).$$

Theorem 5 then yields an isomorphism

$$\mathrm{Sp}_{\Omega}(\mathrm{hInd}(\mathcal{C})) \xrightarrow{\sim} \mathrm{CohTh}^{\Pi}(\mathrm{Ind}(\mathcal{C})), \quad (B_n)_{n\in\mathbb{Z}} \mapsto ([-, B_n])_{n\in\mathbb{Z}}.$$

Indeed, note that both sides are unchanged if we replace $\mathcal{C}$ by its full subcategory generated by suspensions under finite colimits, to which Theorem 5 applies. The canonical functor $\mathrm{hSp}(\mathrm{Ind}(\mathcal{C})) \to \mathrm{Sp}_{\Omega}(\mathrm{hInd}(\mathcal{C}))$ is full, essentially surjective, and conservative, and it identifies parallel morphisms if and only if they are levelwise homotopic. Since levelwise homotopic morphisms are also weakly homotopic, the canonical functor $\mathrm{hSp}(\mathrm{Ind}(\mathcal{C})) \to \mathrm{h}_w\mathrm{Sp}(\mathrm{Ind}(\mathcal{C}))$ factors through $\mathrm{Sp}_{\Omega}(\mathrm{hInd}(\mathcal{C}))$.

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